The conference of young scientists «Pidstryhach readings – 2024»
May 27–29, 2024, Lviv

UDC 512.74

Normal subgroups of iterated wreath products of symmetric groups

Ruslan Skuratovskii[1]

V.I. Vernadsky Taurida National University, Kyiv, Ukraine,
skuratovskii.ruslan@tnu.edu.ua

Normal subgroups and there properties for finite and infinite iterated wreath products $S_{n_1} \wr \ldots \wr S_{n_m}$, $n, m \in \mathbb{N}$ and $A_n \wr S_n$ are founded.

Definition 1. The permutational *subwreath product* $G \wr\wr H$ is the semi-direct product $G \ltimes \tilde{H}^X$, where G acts on the subdirect product [2] $\tilde{H}^X$ by the respective permutations of the subdirect factors. Provided the specification of $\tilde{H}^X$ is established separately.

Definition 2. The set of elements from $S_n \wr S_n, n \geqslant 3$ which presented by the tableaux of form: $[e]_0$, $[a_1, a_2, \ldots, a_n]_1$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_1) = 2k, k \in \mathbb{N}, \tag{1}$$

be called set of type $\widetilde{A}_n^{(1)}$ and denote this set by $e \wr\wr \widetilde{A}_n$. Note that condition (1) uniquely identifies subdirect product.

The set $\widetilde{A}_n^{(1)}$ is subgroup having **normal rank** 2 in $S_n \wr S_n$. We spread this definition on 3-multiple wreath product by recursive way.

Definition 3. *The subgroup* $E \wr \widetilde{A}_n^{(1)}$ *be denoted by* $\widetilde{A}_n^{(2)}$.

Furthermore we prove that $E \wr \widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. The order of $E \wr \widetilde{A}_n^{(2)}$ is $(n!)^{3n} : 2^3$. The subgroup $\widetilde{A}_n^{(1)}$ has **normal rank** 2 in $S_n \wr S_n$.

Definition 4. The set of elements from $S_n \wr S_n \wr S_n, n \geqslant 3$ presented by the tables [1] form:
$[e]_1$, $[e, e, \ldots, e]_2, [a_1, a_2, \ldots, a_n]_3$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_3) = 2k, k \in \mathbb{N}, \tag{2}$$

be denoted by $\widetilde{A}_{n^2}^{(3)}$. Note that condition (2) uniquely identifies subdirect product in $\prod_{i=1}^{n^2} S_n$.

Proposition 1. *The subgroup* $\widetilde{A}_n^{(1)} \lhd S_n \wr S_n$ *as well as* $\widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$.

Definition 5. *A subgroup in* $S_n \wr S_n$ *is called* $\widetilde{T_n}$ *if it consists of:*

1. *elements of* $E \wr A_n$,
2. *elements with the tableau [1] presentation* $[e]_1,\ [\pi_1, \ldots, \pi_n]_2$, *that* $\pi_i \in S_n \setminus A_n$.

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure

$$\tilde{T}_n \simeq (\underbrace{A_n \times A_n \times \cdots \times A_n}_{n}) \rtimes C_2 \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n},$$

where the operation $\boxplus$ *of a subdirect product is subject of item 1) and 2) [3].*

Remark 1. The order of $\widetilde{T_n}$ is $\frac{(n!)^n}{2^{n-1}}$.

Theorem 1. *Proper normal subgroups in* $S_n \wr S_m$, *where* $n, m \geq 3$ *with* $n, m \neq 4$ *are of the following types:*

1. *subgroups that act only on the second level are*

$$E \wr \widetilde{A_m},\ \widetilde{T_m},\ E \wr S_m, E \wr A_n,$$

2. *subgroups that act on both levels are* $A_n \wr \widetilde{A_m}$, $S_n \wr \widetilde{A_m}$, $A_n \wr S_m$,

wherein the subgroup $S_n \wr \widetilde{A_m} \simeq S_n \leftthreetimes (\underbrace{S_m \boxtimes S_m \boxtimes S_m \ldots \boxtimes S_m}_{n})$ *endowed with the subdirect product satisfying to condition* (1).

Theorem 2. *The full list of normal subgroups of* $S_n \wr S_n \wr S_n$ *consists of 50 normal subgroups.*

Література

1. Kaloujnine L. A. Sur les p-group de Sylow. // *C. R. Acad. Sci. Paris.* — 1945. — **221**. — P. 222–224.

2. *Birkhoff, Garrett* (1944), "Subdirect unions in universal algebra Bulletin of the American Mathematical Society, 50 (10): 764-768, doi:10.1090/S0002-9904-1944-08235-9, ISSN 0002-9904, MR 0010542.

3. R. V. Skuratovskii. On commutator subgroups of Sylow 2-subgroups of the alternating group, and the commutator width in wreath products. / Ruslan V. Skuratovskii // European Journal of Mathematics. **vol. 7**: 1. (2021), P. 353-373.