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NEW SHARP BOUNDS FOR THE NORMS OF PROJECTION OPERATORS, DISCRETIZATION AND OPTIMAL SAMPLING RECOVERY

Kateryna Pozharska

Institute of Mathematics of the NAS of Ukraine, Kyiv, Ukraine
Chemnitz University of Technology, Chemnitz, Germany

We show that there are sampling projections onto arbitrary n-dimensional subspaces of the space of bounded functions with at most $2n$ samples and norm of order $\sqrt{n}$. The result is based on a specific type of discretization of the uniform norm which might be of independent interest and is connected to the Marcinkiewicz-Zygmund inequalities.

The theorem of Kadets and Snobar asserts that for any n-dimensional subspace V_n of a normed space G, there is a linear projection $P\colon G \to V_n$ with $\|P\| \le \sqrt{n}$. However, it is not clear what information of $f \in G$ is required to compute its projection.

We consider the case where f is a function and only function evaluations of f are allowed as information. We therefore restrict to $G = B(D)$, i.e., the space of all bounded complex-valued functions on a set D equipped with the sup-norm $\|f\|_\infty := \sup_{x\in D} |f(x)|$.

Theorem 1. *[1] There is an absolute constant $C > 0$ such that the following holds. Let D be a set and V_n be an n-dimensional subspace of $B(D)$. Then there are $2n$ points $x_1, \ldots, x_{2n} \in D$ and functions $\varphi_1, \ldots, \varphi_{2n} \in V_n$ such that $P\colon B(D) \to V_n$ with $Pf = \sum_{i=1}^{2n} f(x_i)\,\varphi_i$ is a projection with $\|P\| \le C\sqrt{n}$.*

Theorem 1 is sharp in the following sense:

- Using $m = \mathcal{O}(n)$ samples, the norm bound $C\sqrt{n}$ cannot be replaced with a lower-order term.
- If we want to use $m = n$ samples, the norm bound $C\sqrt{n}$ has to be replaced by a linear term in n.

Note also, that the oversampling factor 2 in Theorem 1 can be replaced by any constant $c > 1$.

Theorem 1 is based on the following discretization result.

Theorem 2. *[1] There is an absolute constant $C > 0$ such that the following holds. Let D be a set and V_n be an n-dimensional subspace of $B(D)$. Then there are points $x_1, \ldots, x_{2n} \in D$ such that, for all $f \in V_n$, we have*

$$\|f\|_\infty \;\leq\; C \Big(\sum_{i=1}^{2n} |f(x_i)|^2 \Big)^{1/2} \;\leq\; C\, \sqrt{2n} \max_{i=1,\ldots,2n} |f(x_i)|.$$

Further in the talk, we will discuss consequences for optimal recovery in L_p and new sharp bounds for the n-th linear sampling numbers [2].

1. Krieg D., Pozharska K., Ullrich M., Ullrich T., *Sampling projections in the uniform norm.* arXiv: 2401.02220, 2024.

2. Krieg D., Pozharska K., Ullrich M., Ullrich T., *Sampling recovery in L_2 and other norms.* arXiv: 2305.07539, 2023.