The conference of young scientists «Pidstryhach readings – 2024»
May 27–29, 2024, Lviv

UDC 517.54

EXTREMAL PROBLEM ON DOMAINS CONTAINING ELLIPSE POINTS

Iryna Denega and Yaroslav Zabolotnyi

Institute of mathematics of the National Academy of Sciences of Ukraine,
iradenega@gmail.com, yaroslavzabolotnii@gmail.com

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be its one point compactification, U be the open unit disk in $\mathbb{C}$. A function $g_B(z, a)$ which is continuous in $\overline{\mathbb{C}}$, harmonic in $B\backslash\{a\}$ apart from z, vanishes outside B, and in the neighborhood of a has the following asymptotic expansion

$$g_B(z, a) = -\ln|z-a| + \gamma + o(1), \quad z \to a,$$

is called the (classical) Green function of the domain B with pole at $a \in B$. The inner radius $r(B, a)$ of the domain B with respect to a point a is the quantity e^{γ}. Let G be a domain in extended complex plane $\overline{\mathbb{C}}_z$. By a quadratic differential in G we mean the expression $Q(z)dz^2$, where $Q(z)$ is a meromorphic function in G [2].

The following result was established by G.M. Goluzin [1] using the variational method.

Theorem 1. *For functions $f_k(z)$ which univalently map the disc $|z| < 1$ onto mutually non-overlapping domains, $k \in \{1, 2, 3\}$, exact estimate holds*

$$\left|\prod_{k=1}^{3} f_k'(0)\right| \leqslant \frac{64}{81\sqrt{3}}|(f_1(0)-f_2(0))(f_1(0)-f_3(0))(f_2(0)-f_3(0))|.$$

Equality is attained only for functions $w = f_k(z)$ which conformally and univalently map the disc $|z| < 1$ onto the angles $2\pi/3$ with vertex at point $w = 0$ and bisectors of which pass through points $f_k(0)$, $|f_k(0)| = 1$.

E.V. Kostyuchenko (see, for example, [2]) proved that the maximum value of multiplication of inner radiuses for three simply connected non-overlapping domains in the disk is attained for three equal sectors. However, this statement remains valid for multiply connected domains D_k. It follows from V.N. Dubinin's generalization of Theorem 1 inequality to the case of arbitrary meromorphic functions [2].

Using above-posed results, the following theorem is valid.

Let $M = \left\{z = x + iy : \frac{x^2}{d^2} + \frac{y^2}{t^2} = 1,\ d^2 - t^2 = 1\right\}$ and let $d^* = d - \sqrt{d^2-1}$.

Theorem 2. *Let $n \in \mathbb{N}$, $n \geqslant 3$. Then, for any system of different points a_k such that $a_k \in M$, $k = \overline{1, n}$, and for any collection of mutually non-overlapping domains $\{B_k\}_{k=1}^{n}$, $a_k \in B_k \subset \overline{\mathbb{C}} \setminus [-1, 1]$, $k = \overline{1, n}$, the inequality*

$$\prod_{k=1}^{n} r\left(B_k, a_k\right)$$

$$\leqslant \left(\frac{4(d-\sqrt{d^2-1})}{n}\right)^n \left(\frac{1-(d-\sqrt{d^2-1})^n}{1+(d-\sqrt{d^2-1})^n}\right)^n \prod_{k=1}^{n} \left|\frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}}\right|$$

holds. The sign of equality is attained, if a_k and B_k, $k = \overline{1, n}$, are, respectively, the poles and circular domains of the quadratic differential

$$Q(z)dz^2 = -\frac{\left(\frac{z}{2}+\frac{1}{2z}\right)^{n-2}\left(\left(\frac{z}{2}+\frac{1}{2z}\right)^n+1\right)\left(\frac{1}{4}-\frac{1}{2z^2}+\frac{1}{z^4}\right)}{\left(\left(\frac{z}{2}+\frac{1}{2z}\right)^n-(d^*)^n\right)^2\left(1-\left(\frac{z}{2}+\frac{1}{2z}\right)^n(d^*)^n\right)^2}dz^2.$$

Note, that by some linear transformation $w = pz + z_0$ we can transform an arbitrary ellipse $\frac{x-x_0}{d_0^2} + \frac{y-y_0}{t_0^2} = 1$ on the complex plane onto an ellipse of the form $\frac{x^2}{d^2} + \frac{y^2}{t^2} = 1$ for which $d^2 - t^2 = 1$. Moreover, the inner radii of respective domains in this transformation will be treated as $|p| : 1$. Therefore, in order to obtain an estimate of the product of inner radii of non-overlapping domains containing points of an arbitrary ellipse, it is necessary to transform it onto the ellipse M by an appropriate linear transformation and apply Theorem 2.

1. Goluzin G.M. Geometric theory of functions of a complex variable. Amer. Math. Soc. Providence, R.I., 1969.
2. Dubinin V.N. Condenser capacities and symmetrization in geometric function theory. Birkhäuser/Springer, Basel, 2014.
3. Zabolotnii Ya.V., Denega I.V. *An extremal problem on non-overlapping domains containing ellipse points*. Eurasian Math. J. 2021, **12** (4), 82–91.
4. Denega I.V., Zabolotnii Ya.V. *Inequalities for the inner radii of domains containing an arbitrary ellipse points and infinity*. Trends in Mathematics (in print).

ЕКСТРЕМАЛЬНА ЗАДАЧА ДЛЯ ОБЛАСТЕЙ, ЩО МІСТЯТЬ ТОЧКИ ЕЛІПСА

В роботі одержано розв'язок екстремальної задачі про максимум добутку внутрішніх радіусів на системі багатозв'язних областей B_k, $k = \overline{1, n}$, які взаємно не перетинаються, і містять точки a_k, $k = \overline{1, n}$, розташовані на довільному еліпсі $\frac{x^2}{d^2} + \frac{y^2}{t^2} = 1$ для якого $d^2 - t^2 = 1$.