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# Verbal width by set of squares in alternating group $A_n$ and Mathieu groups, criterions of quadraticity in $PSL_2(F_p)$, $GL_2(F_p)$

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The width of a verbal subgroup $V(G, W)$ [1] over a set $W$ is equal to a least value $m \in \mathbb{N} \cup \{\infty\}$ such that every element from $V(G, W)$ can be represented as the product of at most $m$ values of words from $W$.

In a group $G$, the set of squares of its elements is denoted by $\mathbb{S}(G)$. We consider the set of all squares $\mathbb{S}(A_n)$ of the alternating group $A_n$ as a generating set for $A_n$ [2].

**Theorem 1.** *The set of all squares $\mathbb{S}(A_n)$ from $A_n$ does not coincide with the whole alternating group $A_n$ and does not form a proper subgroup of $A_n$. The set $\mathbb{S}(A_n)$ is generating set for $A_n$. The verbal width of $V(A_n, \mathbb{S}(A_n)) = 2$ for $n > 3$.*

**Lemma 1.** *An arbitrary element $g \in A_n$ having the cyclic structure $[(2k)^1, (2r)^1]$ can be presented in the form of a product of 2 squares of $h_1$, $h_2 \in A_n$ with 2 joint letters in their cyclic presentation on $n$-letters alphabet, $2k + 2r - 2$ ways.*

**Theorem 2.** *If in cyclic structure of $g \in A_n$ every even cycle appears an even number of times, i.e. $m_{2k} \equiv 0 \pmod 2$, and at least one of the two following conditions holds:*

*1)*

$$\left[\begin{array}{ll} |Fix\,(g)\,| > 1 & (a) \\ \max\limits_{k \in N}\,(m_{2k-1}) > 1, & (b) \end{array}\right.$$

*2)* $\sum\limits_{l=1}^{L} p_{2l} \equiv 0 \pmod 2$,

*then this $g$ can be presented as $g = h^2$, $h \in A_n$. The vice versa is also true. The condition $m_{2l} \equiv 0 (mod 2)$ is sufficient and necessary in $S_n$.*

**Theorem 3.** *Let $A \in PSL_2(F_p)$, where $\mathbb{F}_p$ is some field, and eigenvalues of $A$ has algebraic multiplicity equal to geometric multiplicity, then for a matrix $A$, there is a matrix $B \in PSL_2(F_p)$ such that*

$$B^2 = A$$

*if and only if,* $tr(A) + 2$ *or* $-tr(A) + 2$ *is a quadratic residue in* $\mathbb{F}_p$.

**Theorem 4.** *Under conditions* $(\frac{\lambda}{p}) = 1$ *in* $F_p$ *and matrix* $A$ *is similar to a Jordan block of the form*

$$J_A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

*i.e.* $A$ *is simple matrix, a square root* $B$ *of* $A$ *exists in* $PSL_2(F_p)$.

The following criterion to be square for semisimple matrix in $GL_2(F_p)$ is found.

**Theorem 5.** *If a matrix* $A \in GL_2(F_p)$ *is semisimple with non multiples eigenvalues, then square root* $\sqrt{A} \in GL_2(F_p)$ *iff* $(\frac{\lambda_i}{p}) = 1$*, where* $1 \le i \le 2$*, in* $F_{p^2}$.

The problem of recognition of square in $S_n$ with using presentation of word in involutive type ray-like generating set $T = \{(1,2); (2,3); \ldots; (n-1, n)\}$ is solved analogous problem for $A_n$ and Mitsuhashi's generating set is considered.

**Theorem 6.** *The verbal width of the verbal subgroups generated by squares of the following* $M_8, M_9, M_{10}$ *Mathieu groups are equal to 1. The structure of* $V(\mathbb{S}(M_9), M_9)$ *is* $(C_3 \times C_3) \rtimes C_2$.

**Theorem 7.** *The following Mathieu groups* $M_{11}$, $M_{12}$, $M_{20}$, $M_{21}$, $M_{22}$, $M_{23}$ *and* $M_{24}$ *have the verbal width* $vw(M_{ij}, \mathbb{S}(M_{ij})) = 2$.

1. *D. Z. Kagan.* The width of verbal subgroups in anomalous products. The width of verbal subgroups in anomalous products. Scientific news. Series Mathematics. Physics. **vol. 255**: 6. Issue 46 (2017), P. 24-29.
2. *R. V. Skuratovskii.* On commutator subgroups of Sylow 2-subgroups of the alternating group, and the commutator width in wreath products. / Ruslan V. Skuratovskii // European Journal of Mathematics. **vol. 7**: 1. (2021), P. 353-373. doi.org/10.1007/s40879-020-00418-9.

## Вербальна ширина над множиною квадратів у знакозмінній групі і групах Матьє, критерій квадратичності в $PSL_2(F_p)$, $GL_2(F_p)$.

*Ukrainian annotation*

Знайдено вербальну ширину по множині квадратів для знакозмінної групи і груп Матьє. Доведені критерії квадратичності для груп $A_n$, $PSL_2(F_p)$ і $GL_2(F_p)$.