The conference of young scientists «Pidstryhach readings – 2022»
May 25–27, 2022, Lviv

UDC 512.74

Normal subgroups of iterated wreath products of symmetric and alternating groups

Ruslan Skuratovskii

National Aviation University, ruslan.skuratovskii@nau.edu.ua

The investigation of invariant subgroups of wreath product have many relations in particular with dynamic systems [1].

The lattice of normal subgroups [2] and their properties for finite iterated wreath products $S_{n_1} \wr \ldots \wr S_{n_m}$, $n, m \in \mathbb{N}$ are found. Special classes of normal subgroups and their orders and generators are found. Further, the monolith of these wreath products has been investigated.

We consider the following generalization of diagonal subgroup of the wreath product base.

Definition 1. A subgroup in $S_n \wr S_n$ is of the type $\widetilde{T_n}$ if it contains:

1. elements of the form $E \wr A_n$,
2. elements with the Kaloujnine tableau [2] presentation $[e]_1,\ [\pi_1 \ldots, \pi_n]_2$, that $\pi_i \in S_n \setminus A_n$.

Definition 2. A subgroup in $S_n \wr S_n \wr S_n$ is of the type $\widetilde{T^{(3)}_{n^2}}$ if it contains:

1. elements of the form $E \wr E \wr A_n$,
2. elements with the tableau [2] presentation $[e]_1,\ [e, \ldots,\ e]_2,\ [\pi_1 \ldots, \pi_{n^2}]_3$, that $\pi_i \in S_n \setminus A_n$.

The minimal number of transpositions in factorization of a permutation corresponding to element $\pi_i \in S_n$ on transposition we will denote by $rnk(\pi_i)$ and call it rank or decrement of permutation. We set $rnk(e) = 0$.

Definition 3. The set of elements from $S_n \wr S_n, n \geqslant 5$ of the form: $[e]_1,\ [a_1, a_2, \ldots, a_n]_2$, satisfying the following condition

$$\sum_{i=1}^{n} rnk([a_i]_2) = 2k, k \in \mathbb{N}, \tag{1}$$

we will call set of type $\widetilde{A}^{(2)}_n$.

Proposition 1. *The set of elements of type* $\widetilde{A_n}^{(2)}$ *forms the subgroup denoted by* $E \wr \widetilde{A_n}$*, furthermore* $E \wr \widetilde{A_n} \lhd S_n \wr S_n$*.*

The ***monolith*** of $S_n \wr S_m$ is $e \wr A_m$.

The subgroup $E \wr E \wr \widetilde{A_n} < S_n \wr S_n \wr S_n$ we denote by $\widetilde{A_n^{(3)}}$.

The set of elements from $S_n \wr S_n \wr S_n, n \geqslant 5$ of the form:
$[e]_1,\ [e, e, \ldots, e]_2, [a_1, a_2, \ldots, a_n]_3$ satisfying the following condition

$$\sum_{i=1}^{n^2} rnk([a_i]_3) = 2k, k \in \mathbb{N}, \tag{2}$$

we denote by $\widetilde{A_{00(n)}^{(n^2)}}$. We note that $E \wr E \wr \widetilde{A_n} < \widetilde{A_{00(n)}^{(n^2)}}$.

Theorem 1. *Proper normal subgroups in* $S_n \wr S_m$ *(active is on the left), where* $n, m \geq 3$ *with* $n, m \neq 4$ *are of the following types:*

1. *subgroups that act only on second level (stabilizing the first level) are*

$$E \wr \widetilde{A_m},\ \widetilde{T_m},\ E \wr S_m,$$

2. *subgroups that acts on both levels are*

$$A_n \wr \widetilde{A_m},\ A_n \wr E,\ S_n \wr \widetilde{A_m},\ A_n \wr S_m,\ S_n \wr E\,.$$

In total there are 8 proper normal subgroups in $S_n \wr S_m$*. Further, all of these groups are splittable groups [2].*

The subgroup $S_n \wr \widetilde{A_m}$ appears only in $S_n \wr S_m$. Thus, the total number of proper normal subgroups in $S_n \wr S_m$, where $n, m \geq 3$ with $n, m \neq 4$ is 8.

The subgroups $A_n \wr \widetilde{A_m} = W'$ [3,4] and $A_n \wr S_m$ are the new normal proper subgroups of $S_n \wr S_m$ relative to the normal subgroups of $S_n \wr A_m$.

Theorem 2. *There are exactly 5 proper normal subgroups in the wreath product* $W = A_n \wr S_n$*.*

Theorem 3. *The normalizer of subgroup* T_n *of type* $\widetilde{T_n}$ *in* $W = S_n \wr S_n$ *is* $N_W(T_n) = S_n \wr A_n$*.*

The lattice of normal subgroups of the finite iterated wreath product $S_{n_1} \wr \ldots \wr S_{n_m}$ is investigated by us too.

The set of normal subgroup of $S_n \wr S_n$ is denoted by $N(S_n \wr S_n)$.

Theorem 4. *The full list of normal subgroups of* $S_n \wr S_n \wr \underset{\frown}{S_n}$ *consists of 50 normal subgroups. The first type* T_{023} *contains:* $E \wr \tilde{A}_n \wr H$, $\widetilde{T_n} \wr H$, *where* $H \in \{\tilde{A}_n, \tilde{A}_{n^2}, S_n\}$. *There are 6 subgroups.*

The second type of subgroups is subclass in T_{023} *with new base of wreath product subgroup* $\tilde{A}_{n^2}$*:* $E \wr S_n \wr \tilde{A}_{n^2}$, $E \wr A_n \wr \tilde{A}_{n^2}$, $E \wr N_i(\underset{\smile}{S_n \wr S_n})$. *Therefore this class has 12 new subgroups. The third type* T_{003}*:* $\widetilde{A_{00(n)}^{(n^2)}} = E \wr \widetilde{E \wr A_n}$, $\widetilde{T_{n^2}^{(3)}}$. *Hence, here are 2 new subgroups.*

The forth type T_{123}*:* $N_i(S_n \wr S_n) \wr S_n$, $N_i(S_n \wr S_n) \wr \tilde{A}_n$ *and* $N_i(S_n \wr S_n) \wr \tilde{A}_{n^2}$. *Thus, there 30 new subgroups in* T_{123}.

1. *Ryan Gopp.* Normal Subgroups of Wreath Product 3-Groups. The University of Akron, Spring 2017, P. 519.

2. *V. I. Sushchansky.* Normal structure of the isometric group of metric spaces of p-adic integers./ V. I. Sushchansky // *Algebraic structures and their application. Kiev,* // Visn of KNU, **vol. 8.** (1988). P. 113–121.

3. *Ruslan V. Skuratovskii, Aled Williams,* Irreducible bases and subgroups of a wreath product in applying to diffeomorphism groups acting on the Mobius band // *Rendiconti del Circolo Matematico di Palermo Series 2* - vol. 70, no. 2, (2021). P. 721–739.

4. *R. V. Skuratovskii.* On commutator subgroups of Sylow 2-subgroups of the alternating group, and the commutator width in wreath products. / Ruslan V. Skuratovskii // European Journal of Mathematics. **vol. 7**: 1. (2021), P. 353-373. doi.org/10.1007/s40879-020-00418-9.

Нормальні підгрупи ітерованого вінцевого добутку симетричних і знакозмінних груп підстановок.

Ukrainian annotation

Знайдено решітку нормальних підгруп ітерованого вінцевого добутку $S_{n_1} \wr \ldots \wr S_{n_k}$. Описано всі нормальні підгрупи. Доведено монолітичність даної групи.