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Verbal subgroups of alternating group A_n and Matieu groups

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The verbal width of a free group was investigated by Sucharit Sarkar [1] Sarkar proved that an arbitrary commutator of free group of rank greater than 1 cannot be generated by only 2 squares.

We consider set of all squares $S(A_n)$ of A_n as generating set for A_n.

The **width of the verbal subgroup** $V(G, W)$ over set W is equal to a least value of $m \in \mathrm{N} \cup \{\infty\}$ such that every element of the subgroup $V(G, W)$ is represented as the product of at most m values of words from W.

The conditions when an arbitrary $g \in A_n$ as well as $h \in PSL_2(F_p)$ can be presented as one squares were also found by us.

Therefore, we research the verbal width by squares of A_n and some Matieu groups. In a group G a **set of squares of its elements** is denoted by $\mathbb{S}(\mathbb{G})$.

Since A_n is generated by all pairs of transpositions in particular by Mitsuhashi's generating set [2] then $S(A_n)$ generates whole A_n. Therefore one can consider $diam(A_n, S(A_n))$ [6]. Thus it can be applied to investigation of distance between permutations on Cayley graph.

Thus, problem of element verbal width is solved for the verbal subgroup $V(S(A_n), A_n)$ which coincides with A_n.

Theorem 1. *The set of all squares $S(A_n)$ from A_n does not coincide with the whole alternating group A_n and does not form a proper subgroup of A_n. The normal closure of $S(A_n)$ is A_n.*

Theorem 2. *An arbitrary element g of A_n having cyclic structure $[(2k)^{m_{2k}}, (2r)^{m_{2r}}]$ can be presented in the form of a product of 2 squares of permutations of A_n with 2 joint elements.*

Theorem 3. *The verbal width of $V(A_n, S(A_n))$ in A_n is 2.*

Corollary 1. *An arbitrary element of A_n can be presented in the form of a product of two squares of elements from A_n.*

Corollary 2. *An arbitrary element g of A_n having cyclic structure $[(2k), (2r)]$ can be presented in form of a product of 2 squares by $2k + 2r - 2$ ways.*

The conditions when an arbitrary $g \in A_n$ can be presented as one squares were also found by us. For instance the element $g = (10, 11, 12)(1, 2, 3)(4, 5)(6, 7)$ is square in A_{12}.

Proposition 1. *Let the element $\pi = (a_1 \dots a_l)\,(b_1 \dots b_l) \in A_{2l}$ then the square roots from π exist in A_{2l}. Moreover, there are $l+1$ different roots from π when l is odd and there l different roots if l is even.*

For instance, the permutation consisting of two 3-cycles $s = (123)(456)$ can be presented as square of the following 4 elements: $t_1 = (142536)$, $t_2 = (152634)$, $t_3 = (162435)$, $t_4 = (132)(465)$.

Theorem 4. *The verbal width of verbal subgroups generated by squares of the following Mathieu groups M_8, M_9, M_{10} are equal to 1. The structure of $H_9 = V(S(M_9), M_9)$ is $H_9 \simeq (C_3 \times C_3) \rtimes C_2$.*

Theorem 5. *The following prime Mathieu groups M_{11}, M_{12}, M_{20}, M_{21} and M_{22} have verbal width $vw(M_{ij}, S(M_{ij})) = 2$ by square 2.*

Theorem 6. *Let $A \in PSL_2(F_p)$, where $\mathbb{F}_p$ is some field. For matrix $A \in PSL_2(F_p)$, there is a matrix B over $\mathbb{F}_p$ such that*

$$B^2 = A$$

if and only if, $tr(A) + 2$ or $-tr(A) + 2$ is a quadratic residue in $\mathbb{F}_p$.

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Вербальні підгрупи знакозмінної групи A_n і груп Матьє.

Досліджено вербальну ширину вербальних підгруп в знакозмінній групі і в деяких групах Матьє.