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A CLASSIFICATION OF QUASIGROUPS ACCORDING TO THE SETS OF TRANSLATIONS

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An algebra $(Q;\cdot;\overset{\ell}{\cdot};\overset{r}{\cdot})$ is called a *quasigroup*, if $(\cdot)$ is an invertible operation and $(\overset{\ell}{\cdot})$ and $(\overset{r}{\cdot})$ are its left and right inverses, i.e. the identities

$$(x\overset{\ell}{\cdot}y)\cdot y=x,\quad (x\cdot y)\overset{\ell}{\cdot}y=x,\quad x\cdot(x\overset{r}{\cdot}y)=y,\quad x\overset{r}{\cdot}(x\cdot y)=y$$

hold. σ-parastrophe $(\overset{\sigma}{\cdot})$ of $(\cdot)$ is defined by

$$x_{1\sigma}\overset{\sigma}{\cdot}x_{2\sigma}=x_{3\sigma}\ \Leftrightarrow\ x_1\cdot x_2=x_3,\quad \sigma\in S_3:=\{\iota,\ell,s,r,\ell s,\ell r\},$$

where $\ell:=(13)$, $s:=(12)$, $r:=(23)$. Bijections L_a, R_a, M_a of a quasigroup $(Q;\cdot)$ are called *left*, *right* and *middle translations* [1] and are defined by

$$L_a(x):=a\cdot x,\qquad R_a(x):=x\cdot a,\qquad M_a(x):=x\overset{r}{\cdot}a.$$

Every element a of a quasigroup $(Q;\cdot)$ defines six bijections:

$$\mathscr{M}_a:=\{M_a,M_a^{-1},L_a,L_a^{-1},R_a,R_a^{-1}\}.\tag{1}$$

The definition of the *σ-parastrophe* of left, right and middle translation are:

$${}^{\sigma}L_a(x):=a\overset{\sigma^{-1}}{\cdot}x,\qquad {}^{\sigma}R_a(x):=x\overset{\sigma^{-1}}{\cdot}a,\qquad {}^{\sigma}M_a(x):=x\overset{r\sigma^{-1}}{\cdot}a,$$

where $\sigma\in S_3$. Moreover, the following relationships are true:

$${}^{\tau}({}^{\sigma}L_a)={}^{\tau\sigma}L_a,\qquad {}^{\tau}({}^{\sigma}R_a)={}^{\tau\sigma}R_a,\qquad {}^{\tau}({}^{\sigma}M_a)={}^{\tau\sigma}M_a.$$

Proposition 1. In all parastrophes of a quasigroup $(Q;\cdot)$ an arbitrary element a defines the same set of translations $\mathscr{M}_a$ (see (1)):

$$\mathscr{M}_a=\{{}^{\sigma}L_a\mid\sigma\in S_3\}=\{{}^{\sigma}R_a\mid\sigma\in S_3\}=\{{}^{\sigma}M_a\mid\sigma\in S_3\}.$$

Proposition 2. Each translation is a σ-parastrophe of the middle translation for some $\sigma \in S_3$:

$$M_a = {}^{\iota}M_a, \qquad L_a = {}^{\ell s}M_a, \qquad R_a = {}^{r}M_a,$$

$$M_a^{-1} = {}^{s}M_a, \qquad L_a^{-1} = {}^{\ell}M_a, \qquad R_a^{-1} = {}^{rs}M_a.$$

The permutation σ is called *direction* of the translation ${}^{\sigma}M_a$.

The *σ-direction set of translations*, i.e. the set of all translations of the direction σ of a quasigroup $(Q;\cdot)$ is defined by

$${}^{\sigma}\mathcal{M} := \{{}^{\sigma}M_x \mid x \in Q\}, \quad \sigma \in S_3.$$

Theorem 1. *All equalities of two translation sets of different directions determine the following classes of quasigroups:*

${}^{\ell}\mathcal{M} = {}^{\ell s}\mathcal{M}$	$L_x^{-1} = L_{\alpha(x)}$	$\alpha(x) \cdot xy = y$	*LIP quasigroup*
${}^{r}\mathcal{M} = {}^{rs}\mathcal{M}$	$R_x^{-1} = R_{\alpha(x)}$	$yx \cdot \alpha(x) = y$	*RIP quasigroup*
${}^{\iota}\mathcal{M} = {}^{s}\mathcal{M}$	$M_x^{-1} = M_{\alpha(x)}$	$yz = \alpha(zy)$	*MIP quasigroup*
${}^{\ell}\mathcal{M} = {}^{r}\mathcal{M}$ ${}^{rs}\mathcal{M} = {}^{\ell s}\mathcal{M}$	$L_x^{-1} = R_{\alpha(x)}$ $R_x^{-1} = L_{\alpha(x)}$	$xy \cdot \alpha(x) = y$	*CIP quasigroup*
${}^{\ell}\mathcal{M} = {}^{\iota}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{\ell s}\mathcal{M}$	$L_x^{-1} = M_{\alpha(x)}$ $M_x^{-1} = L_{\alpha(x)}$	$xy \cdot y = \alpha(x)$	$\mathfrak{K}_1$
${}^{\iota}\mathcal{M} = {}^{rs}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{r}\mathcal{M}$	$R_x^{-1} = M_{\alpha(x)}$ $M_x^{-1} = R_{\alpha(x)}$	$yx \cdot y = \alpha(x)$	$\mathfrak{K}_2$
${}^{\ell s}\mathcal{M} = {}^{\iota}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{\ell}\mathcal{M}$	$L_x = M_{\alpha(x)}$ $M_x^{-1} = L_{\alpha(x)}^{-1}$	$y \cdot xy = \alpha(x)$	$\mathfrak{K}_3$
${}^{r}\mathcal{M} = {}^{\iota}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{rs}\mathcal{M}$	$R_x = M_{\alpha(x)}$ $M_x^{-1} = R_{\alpha(x)}^{-1}$	$y \cdot yx = \alpha(x)$	$\mathfrak{K}_4$
${}^{\ell s}\mathcal{M} = {}^{r}\mathcal{M}$ ${}^{\ell}\mathcal{M} = {}^{rs}\mathcal{M}$	$L_x = R_{\alpha(x)}$ $R_x^{-1} = L_{\alpha(x)}^{-1}$	$\alpha(x) \cdot y = y \cdot x$	$\mathfrak{K}_5$

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КЛАСИФІКАЦІЯ КВАЗІГРУП ВІДПОВІДНО ДО МНОЖИН ТРАНСЛЯЦІЙ

Введено поняття напрямку трансляції та напрямку множини трансляцій. Знайдено класи квазігруп, в кожній з яких дві множини різних напрямків збігаються. Встановлено, що всього існує дев'ять таких класів. Всі ці класи є добре відомими многовидами квазігруп з деякими властивостями оборотності, серед яких IP-квазігрупи, CIP-квазігрупи.