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CLASSIFICATION OF SYMMETRY REDUCTIONS FOR THE EIKONAL EQUATION

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We present the results concerning the relationship between the structural properties of low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$ and the properties of the reduced equations for the eikonal equation. To obtain those results, we have performed the classification of the invariants as well as of the ansatzes for the above mentioned subalgebras. We also present some classes of invariant solutions for the eikonal equation.

The book is intended for specialists in the theory of Lie algebras, theory of differential equations, theoretical and mathematical physics, and mechanics. Ref. 104.

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У монографії представлено результати вивчення взаємозв'язку між структурними властивостями низькорозмірних ($dimL \leq 3$) неспряжених підалгебр алгебри Лі групи Пуанкаре $P(1,4)$ та властивостями редукованих рівнянь для рівняння ейконала. Результати ґрунтуються на проведеній попередньо класифікації інваріантів, а також анзаців для вищезгаданих підалгебр. Представлено також деякі класи інваріантних розвязків для рівняння ейконала.

Для спеціалістів з теорії алгебр Лі, теорії диференціальних рівнянь, теоретичної і математичної фізики та механіки. Бібліогр. 104 назв.

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Preface

It is well known that mathematical models of the real processes of nature can be very often described with the help of partial differential equations (PDEs).

It is also known that the important PDEs of the theoretical and mathematical physics, mechanics, gas dynamics, etc. have non-trivial symmetry groups. Many of the scientists used and use this fact to investigate those types of PDEs. At present, many scientific works have been published on this topic. It is impossible to present here even the majority of them. Therefore, we only mention here some books concerning this topic [1–27] (see also the references therein).

In the following, we focus our attention on some applications of the classical Lie method to investigate PDEs with non-trivial symmetry groups.

In 1895, Lie [28] considered solutions invariant with respect to groups admitted by the higher-order PDEs.

It turned out that the problem of symmetry reduction and the construction of classes of independent invariant solutions for PDEs with non-trivial symmetry groups was reduced to a pure algebraic problem of describing all nonconjugate (non-similar) subalgebras of the Lie algebras of symmetry groups of those equations. The details can be found in [6,8,10,14,29,30] (see also the references therein).

In 1975, Patera, Winternitz, and Zassenhaus [31] proposed a general method for describing the nonconjugate subalgebras of Lie algebras with nontrivial ideals.

Two years ago, Patera and Winternitz [32] described nonconjugate subalgebras of real three- and four-dimensional Lie algebras.

The results of those two works make it possible to describe a subgroup structure of the symmetry groups as well as to construct classes of invariant solutions for many PDEs.

However, it turned out that the reduced equations, obtained with the help of nonconjugate subalgebras of the same ranks of the Lie algebras of the symmetry groups of some PDEs, were of different types. The details on this theme can be found in [33–42] (see also the references therein).

Grundland, Harnad, and Winternitz [33] were the first to point out and investigate the similar phenomenon.

The results obtained cannot be explained within the frame of usual approach. It means that when using only the rank of nonconjugate subalgebras of the Lie algebras of the symmetry groups of some PDEs under investigation, we cannot explain differences in the properties of their reduced equations, which are obtained using nonconjugate subalgebras of the same ranks of the Lie algebras of the symmetry groups of those PDEs.

It is well known that the nonconjugate subalgebras of the same rank of the Lie algebras can have different structural properties. Therefore, in order to try to explain some of the differences in the properties of the reduced equations for PDEs with nontrivial symmetry groups, we suggest to investigate the relationship between the structural properties of nonconjugate subalgebras of the same rank of the Lie algebras of the symmetry groups of those PDEs and properties of the reduced equations corresponding with them [43].

In our monograph, among other things, we present the results concerning the realization of our suggestion for the eikonal equation. More detailed, we present the results concerning the relationship between the structural properties of low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$ [44] and the properties of the reduced equations for the eikonal equation. We also present some classes of invariant solutions for the eikonal equation.

In Chapter 1, we present the results of classification of functional bases of invariants for one-, two-, and three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$.

In Chapter 2, we present the results of the classification of anzatzes for the eikonal equation. The results are obtained using the results of the classification of functional bases of invariants for low-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$ (see Chapter 1).

In Chapter 3, we present the results of the classification of symmetry reductions for the eikonal equation. The results are obtained using the results of the classification of ansatzes for the eikonal equation (see Chapter 2). Some classes of the invariant solutions for the equation under consideration are also presented.

Chapter 1

Classification of invariants for some nonconjugate subgroups of the Poincaré group $P(1,4)$

In this chapter, we present the results of the classification of functional bases of invariants in space $M(1,3) \times R(u)$ for all nonconjugate subalgebras of dimensions 1, 2, and 3 of the Lie algebra of the Poincaré group $P(1,4)$. Here, and in what follows, $M(1,3)$ is the four-dimensional Minkowski space, $R(u)$ is the real number axis of the dependent variable u.

The results are obtained using structural properties of low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ [44].

1.1 Lie algebra of the Poincaré group $P(1,4)$ and its nonconjugate subalgebras

The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. It is the smallest group, which contains, as subgroups, the extended Galilei group $\widetilde{G}(1,3)$ [45] (the symmetry group of classical physics) and the Poincaré group $P(1,3)$ (the symmetry group of relativistic physics). The group $P(1,4)$ has a wide applications in theoretical and mathematical physics (see, for example, [17, 46–52]).

Lie algebra of the group $P(1,4)$ is generated by 15 bases elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu, \nu = 0, 1, 2, 3, 4)$ and P_μ $(\mu = 0, 1, 2, 3, 4)$, which satisfy the commutation relations

$$[P_\mu, P_\nu] = 0, \qquad [M_{\mu\nu}, P_\sigma] = g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu,$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma} M_{\nu\rho} + g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$.

Consider the following representation [53–55] of the Lie algebra of the group $P(1,4)$:

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_1 = -\frac{\partial}{\partial x_1}, \quad P_2 = -\frac{\partial}{\partial x_2}, \quad P_3 = -\frac{\partial}{\partial x_3},$$
$$P_4 = -\frac{\partial}{\partial u}, \quad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \; x_4 \equiv u.$$

In the following, we will use the next bases elements:

$$G = M_{04}, \quad L_1 = M_{23}, \quad L_2 = -M_{13}, \quad L_3 = M_{12},$$

$$P_a = M_{a4} - M_{0a}, \quad C_a = M_{a4} + M_{0a}, \quad (a = 1, 2, 3),$$

$$X_0 = \frac{1}{2}\left(P_0 - P_4\right), \; X_k = P_k \; (k = 1, 2, 3), \; X_4 = \frac{1}{2}\left(P_0 + P_4\right).$$

Nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ have been described in the papers [56–60].

Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ is generated by the following bases elements:

$$L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4.$$

In this chapter, we use the full list of the nonconjugate (up to $P(1,4)$ - conjugation) subalgebras of the Lie algebra of the group $P(1,4)$, which can be found in [18].

By now, we performed classification of all nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of dimensions ≤ 5 [44, 61, 62] using Mubarakzyanov's classification obtained in [63, 64].

1.2 Classification of functional bases of invariants for one-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$

In this section, we present the results of the classification of functional bases of invariants in the space $M(1,3) \times R(u)$ for all one-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$.

It is known that is only one type of one-dimensional real Lie algebras [63]. We denote it by A_1 [65]. Since all one-dimensional Lie algebras are isomorphic, they are of the type A_1.

The results of the classification of one-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ can be formulated as

Proposition. *The Lie algebra of the group $P(1,4)$ contains* 20 *one-dimensional nonconjugate subalgebras of the type* A_1.

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\langle G \rangle$:

   $$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3, \quad \omega_4 = (x_0^2 - u^2)^{1/2} \ ;$$

2. $\langle G + \alpha X_1, \ \alpha > 0 \rangle$:

   $$\omega_1 = x_1 - \alpha \ln(x_0 + u), \quad \omega_2 = x_2, \quad \omega_3 = x_3,$$
   $$\omega_4 = (x_0^2 - u^2)^{1/2} \ ;$$

3. $\left\langle L_3 + \frac{1}{2}(P_3 + C_3) \right\rangle$:

   $$\omega_1 = x_0, \quad \omega_2 = x_1^2 + x_2^2, \quad \omega_3 = x_3^2 + u^2,$$
   $$\omega_4 = \arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u} \ ;$$

4. $\langle L_3 + \lambda G, \ \lambda > 0 \rangle$:

   $$\omega_1 = x_3, \quad \omega_2 = (x_0^2 - u^2)^{1/2}, \quad \omega_3 = (x_1^2 + x_2^2)^{1/2},$$
   $$\omega_4 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} \ ;$$

5. $\left\langle L_3 + \frac{\lambda}{2}(P_3 + C_3), \ 0 < \lambda < 1 \right\rangle$:

   $$\omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = (x_3^2 + u^2)^{1/2},$$
   $$\omega_4 = \lambda \arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u} \ ;$$

6. $\langle L_3 + \lambda G + \alpha X_3, \ \lambda > 0, \ \alpha > 0 \rangle$:

   $$\omega_1 = (x_0^2 - u^2)^{1/2}, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$

$$\omega_3 = \alpha \ln(x_0+u) - \lambda x_3, \quad \omega_4 = x_3 + \alpha \arctan \frac{x_1}{x_2} \ ;$$

7. $\left\langle L_3 + \frac{\lambda}{2}(P_3 + C_3) + \alpha(X_0 + X_4),\ 0 < \lambda < 1, \alpha > 0 \right\rangle$:

$$\omega_1 = x_1^2 + x_2^2, \quad \omega_2 = x_3^2 + u^2, \quad \omega_3 = x_0 - \alpha \arctan \frac{x_1}{x_2},$$

$$\omega_4 = \lambda x_0 - \alpha \arctan \frac{x_3}{u} \ ;$$

8. $\left\langle L_3 + \frac{1}{2}(P_3 + C_3) + \alpha(X_0 + X_4),\ \alpha > 0 \right\rangle$:

$$\omega_1 = x_1^2 + x_2^2, \quad \omega_2 = x_3^2 + u^2, \quad \omega_3 = x_0 - \alpha \arctan \frac{x_1}{x_2},$$

$$\omega_4 = x_0 - \alpha \arctan \frac{x_3}{u} \ .$$

It should be noted that next functional bases are invariant with respect to corresponding nonconjugate subalgebras of the type A_1 of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

9. $\langle L_3 \rangle$:

$$\omega_1 = x_0, \quad \omega_2 = x_3, \quad \omega_3 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_4 = u \ ;$$

10. $\langle L_3 - P_3 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$

$$\omega_3 = (x_0^2 - x_3^2 - u^2)^{1/2}, \quad \omega_4 = \arctan \frac{x_1}{x_2} + \frac{x_3}{x_0 + u} \ ;$$

11. $\langle L_3 + 2X_4 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$

$$\omega_3 = x_3, \quad \omega_4 = x_0 - u + 2\arctan\frac{x_2}{x_1}\ ;$$

12. $\langle L_3 - P_3 + 2\alpha X_0,\ \alpha > 0\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = 2\alpha\arctan\frac{x_1}{x_2} - x_0 - u,$$

$$\omega_3 = (x_0 + u)^2 + 4x_3\alpha,$$

$$\omega_4 = 2(x_0 + u)^3 + 12\alpha^2(x_0 - u) + 12\alpha x_3(x_0 + u)\ ;$$

13. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0\rangle$:

$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = x_0 - \alpha\arctan\frac{x_1}{x_2},$$

$$\omega_4 = u\ ;$$

14. $\langle L_3 + \alpha X_3,\ \alpha > 0\rangle$:

$$\omega_1 = x_0, \quad \omega_2 = u, \quad \omega_3 = x_1^2 + x_2^2,$$

$$\omega_4 = x_3 + \alpha\arctan\frac{x_1}{x_2}\ ;$$

15. $\langle P_3\rangle$:

$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_0 + u,$$

$$\omega_4 = (x_0^2 - x_3^2 - u^2)^{1/2}\ ;$$

16. $\langle P_3 - 2X_0\rangle$:

$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = (x_0 + u)^2 + 4x_3,$$

$$\omega_4 = x_0 - u + \frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u)\ ;$$

17. $\langle P_3 - X_1\rangle$:

$$\omega_1 = x_2, \quad \omega_2 = x_0 + u, \quad \omega_3 = x_1(x_0 + u) - x_3,$$

$\omega_4 = x_3^2 + 2u(x_0 + u)$;

18. $\langle X_0 + X_4 \rangle$:

$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3, \quad \omega_4 = u$;

19. $\langle X_4 \rangle$:

$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3, \quad \omega_4 = x_0 + u$;

20. $\langle X_4 - X_0 \rangle$:

$\omega_1 = x_0, \quad \omega_2 = x_1, \quad \omega_3 = x_2, \quad \omega_4 = x_3$.

1.3 Classification of functional bases of invariants for two-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$

In this section, we present the results of the classification of functional bases of invariants in the space $M(1,3) \times R(u)$ for all two-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$.

It is known that there are only two different types of the real two-dimensional Lie algebras: decomposable $A_1 \oplus A_1 \equiv 2A_1$ and indecomposable A_2 [63]. Lie algebras of the type $2A_1$ are Abelian.

Bases elements (e_1 and e_2) of Lie algebras of the type A_2 satisfy the commutation relations: $[e_1, e_2] = e_2$ [66]. Lie algebras of the type A_2 are solvable [63, 66].

Below, we present the results obtained.

1.3.1 Lie Algebras of the Type $2A_1$

The results of the classification of two-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ can be formulated as

Proposition. *The Lie algebra of the group $P(1,4)$ contains 42 two-dimensional nonconjugate subalgebras of the type $2A_1$.*

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\langle G\rangle \oplus \langle L_3\rangle$:

   $$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = (x_0^2 - u^2)^{1/2} \ ;$$

2. $\langle G\rangle \oplus \langle X_1\rangle$:

   $$\omega_1 = x_2, \quad \omega_2 = x_3, \quad \omega_3 = (x_0^2 - u^2)^{1/2} \ ;$$

3. $\langle G + \alpha X_2,\ \alpha > 0\rangle \oplus \langle X_1\rangle$:

   $$\omega_1 = x_3, \quad \omega_2 = (x_0^2 - u^2)^{1/2}, \quad \omega_3 = x_2 - \alpha \ln(x_0 + u) \ ;$$

4. $\langle G + \alpha X_3,\ \alpha > 0\rangle \oplus \langle L_3\rangle$:

   $$\omega_1 = x_3 - \alpha \ln(x_0 + u), \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$
   $$\omega_3 = (x_0^2 - u^2)^{1/2} \ ;$$

5. $\langle G\rangle \oplus \langle L_3 + \alpha X_3,\ \alpha > 0\rangle$:

   $$\omega_1 = x_3 + \alpha \arctan \frac{x_1}{x_2}, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$

$$\omega_3 = (x_0^2 - u^2)^{1/2} \ ;$$

6. $\langle G + \alpha X_3,\ \alpha > 0\rangle \oplus \langle L_3 + \beta X_3,\ \beta > 0\rangle :$

$$\omega_1 = x_3 - \alpha \ln(x_0 + u) - \beta \arctan \frac{x_2}{x_1},$$
$$\omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = (x_0^2 - u^2)^{1/2} \ ;$$

7. $\langle L_3\rangle \oplus \langle P_3 + C_3\rangle :$

$$\omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = (x_3^2 + u^2)^{1/2} \ ;$$

8. $\langle L_3 + \lambda G,\ \lambda > 0\rangle \oplus \langle X_3\rangle :$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0^2 - u^2)^{1/2},$$
$$\omega_3 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} \ ;$$

9. $\left\langle L_3 + \frac{1}{2}(P_3 + C_3)\right\rangle \oplus \langle X_0 + X_4\rangle :$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_3^2 + u^2)^{1/2},$$
$$\omega_3 = \arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u} \ ;$$

10. $\left\langle L_3 + \frac{\lambda}{2}(P_3 + C_3),\ 0 < \lambda < 1\right\rangle \oplus \langle X_0 + X_4\rangle :$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_3^2 + u^2)^{1/2},$$
$$\omega_3 = \lambda \arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u} \ ;$$

11. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0\rangle \oplus \langle P_3 + C_3\rangle :$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_3^2 + u^2)^{1/2},$$

$$\omega_3 = x_0 - \alpha \arctan \frac{x_1}{x_2} \ ;$$

12. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0\rangle \oplus \langle P_3 + C_3 + 2\beta(X_0 + X_4),$ $\beta > 0\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_3^2 + u^2)^{1/2},$$
$$\omega_3 = \beta \arctan \frac{x_3}{u} - x_0 + \alpha \arctan \frac{x_1}{x_2} \ .$$

It should be noted that next functional bases are invariant with respect to corresponding nonconjugate subalgebras of the type $2A_1$ of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

13. $\langle L_3\rangle \oplus \langle P_3\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$
$$\omega_3 = (x_0^2 - x_3^2 - u^2)^{1/2} \ ;$$

14. $\langle L_3 - P_3\rangle \oplus \langle X_4\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$
$$\omega_3 = \arctan \frac{x_1}{x_2} + \frac{x_3}{x_0 + u} \ ;$$

15. $\langle L_3\rangle \oplus \langle X_4\rangle$:

$$\omega_1 = x_3, \quad \omega_2 = x_0 + u, \quad \omega_3 = (x_1^2 + x_2^2)^{1/2} \ ;$$

16. $\langle L_3\rangle \oplus \langle X_0 + X_4\rangle$:

$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = u \ ;$$

17. $\langle L_3\rangle \oplus \langle X_4 - X_0\rangle$:

$$\omega_1 = x_0, \quad \omega_2 = x_3, \quad \omega_3 = (x_1^2 + x_2^2)^{1/2} \ ;$$

18. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0\rangle \oplus \langle X_4\rangle :$

$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$
$$\omega_3 = x_0 + u - \alpha \arctan \frac{x_1}{x_2} \ ;$$

19. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0\rangle \oplus \langle X_4 - X_0\rangle :$

$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = x_0 - \alpha \arctan \frac{x_1}{x_2} \ ;$$

20. $\langle L_3 + \alpha X_3,\ \alpha > 0\rangle \oplus \langle X_4\rangle :$

$$\omega_1 = x_0 + u,$$
$$\omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = x_3 + \alpha \arctan \frac{x_1}{x_2} \ ;$$

21. $\langle L_3 + \alpha X_3,\ \alpha > 0\rangle \oplus \langle X_0 + X_4\rangle :$

$$\omega_1 = u,$$
$$\omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = x_3 + \alpha \arctan \frac{x_1}{x_2} \ ;$$

22. $\langle L_3 + \alpha X_3,\ \alpha > 0\rangle \oplus \langle X_4 - X_0\rangle :$

$$\omega_1 = x_0,$$
$$\omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = x_3 + \alpha \arctan \frac{x_1}{x_2} \ ;$$

23. $\langle L_3 + 2X_4\rangle \oplus \langle X_3\rangle :$

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$
$$\omega_3 = x_0 - u + 2 \arctan \frac{x_2}{x_1} \ ;$$

24. $\langle L_3 - P_3 + 2\alpha X_0,\ \alpha \neq 0\rangle \oplus \langle X_4\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_0 + u - 2\alpha \arctan \frac{x_1}{x_2},$$
$$\omega_3 = (x_0 + u)^2 + 4x_3\alpha \ ;$$

25. $\langle L_3 + 2X_4\rangle \oplus \langle P_3 - 2\beta X_0,\ \beta > 0\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0 + u)^2 + 4\beta x_3,$$
$$\omega_3 = 4\beta \arctan \frac{x_1}{x_2} - 2\beta(x_0 - u) - \frac{1}{3\beta}(x_0 + u)^3 -$$
$$-2x_3(x_0 + u) \ ;$$

26. $\langle L_3 + 2X_4\rangle \oplus \langle P_3\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_0 + u,$$
$$\omega_3 = \frac{x_3^2}{x_0 + u} + 2 \arctan \frac{x_1}{x_2} + 2u \ ;$$

27. $\langle L_3\rangle \oplus \langle P_3 - 2X_0\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0 + u)^2 + 4x_3,$$
$$\omega_3 = x_0 - u + \frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) \ ;$$

28. $\langle P_1\rangle \oplus \langle P_2\rangle$:

$$\omega_1 = x_3, \quad \omega_2 = x_0 + u, \quad \omega_3 = (x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} \ ;$$

29. $\langle P_1 - X_3\rangle \oplus \langle P_2\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2},$$
$$\omega_3 = x_3 - \frac{x_1}{x_0 + u} \ ;$$

30. $\langle P_1 - X_3\rangle \oplus \langle P_2 - \gamma X_2 - \beta X_3,\ \beta > 0, \gamma > 0\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = \frac{\beta x_2}{x_0 + u + \gamma} + \frac{x_1}{x_0 + u} - x_3,$$

$$\omega_3 = \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \gamma} + 2u\ ;$$

31. $\langle P_1 - X_3\rangle \oplus \langle P_2 - \gamma X_2,\ \gamma > 0\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = \frac{x_1}{x_0 + u} - x_3,$$

$$\omega_3 = \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \gamma} + 2u\ ;$$

32. $\langle P_1\rangle \oplus \langle P_2 - X_2 - \beta X_3,\ \beta > 0\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_3 - \frac{\beta x_2}{x_0 + u + 1},$$

$$\omega_3 = \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + 1} + 2u\ ;$$

33. $\langle P_1\rangle \oplus \langle P_2 - X_2\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_3,$$

$$\omega_3 = \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + 1} + 2u\ ;$$

34. $\langle P_3\rangle \oplus \langle X_1\rangle$:

$$\omega_1 = x_2, \quad \omega_2 = x_0 + u, \quad \omega_3 = (x_0^2 - x_3^2 - u^2)^{1/2}\ ;$$

35. $\langle P_3\rangle \oplus \langle X_4\rangle$:

$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_0 + u\ ;$$

36. $\langle P_3 - X_1\rangle \oplus \langle X_4\rangle$:

$$\omega_1 = x_2, \quad \omega_2 = x_0 + u, \quad \omega_3 = x_1 - \frac{x_3}{x_0+u} \ ;$$

37. $\langle P_3 - X_2\rangle \oplus \langle X_1\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_0^2 - x_3^2 - u^2)^{1/2},$$
$$\omega_3 = x_2 - \frac{x_3}{x_0+u} \ ;$$

38. $\langle P_3 - 2X_0\rangle \oplus \langle X_4\rangle$:

$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = (x_0+u)^2 + 4x_3 \ ;$$

39. $\langle P_3 - 2X_0\rangle \oplus \langle X_1\rangle$:

$$\omega_1 = x_2, \quad \omega_2 = (x_0+u)^2 + 4x_3,$$
$$\omega_3 = x_0 - u + \frac{1}{6}(x_0+u)^3 + x_3(x_0+u) \ ;$$

40. $\langle X_0 + X_4\rangle \oplus \langle X_4 - X_0\rangle$:

$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3 \ ;$$

41. $\langle X_1\rangle \oplus \langle X_4\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_2, \quad \omega_3 = x_3 \ ;$$

42. $\langle X_1\rangle \oplus \langle X_4 - X_0\rangle$:

$$\omega_1 = x_0, \quad \omega_2 = x_2, \quad \omega_3 = x_3 \ .$$

1.3.2 Lie Algebras of the Type A_2

$[e_1, e_2] = e_2$

The results of the classification of two-dimensional nonconjugate subalgebras of the type A_2 of the Lie algebra of the group $P(1,4)$ can be formulated as

Proposition. *The Lie algebra of the group $P(1,4)$ contains seven two-dimensional nonconjugate subalgebras of the type A_2.*

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\langle -G,\ P_3\rangle :$

   $$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = (x_0^2 - x_3^2 - u^2)^{1/2} \ ;$$

2. $\langle -G,\ X_4\rangle :$

   $$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3 \ ;$$

3. $\langle -G - \alpha X_1,\ X_4,\ \alpha > 0\rangle :$

   $$\omega_1 = x_2, \quad \omega_2 = x_3, \quad \omega_3 = x_1 - \alpha \ln(x_0 + u) \ ;$$

4. $\langle -G - \alpha X_1,\ P_3,\ \alpha > 0\rangle :$

   $$\omega_1 = x_1 - \alpha \ln(x_0 + u), \quad \omega_2 = x_2,$$
   $$\omega_3 = (x_0^2 - x_3^2 - u^2)^{1/2} \ ;$$

5. $\left\langle -\frac{1}{\lambda} L_3 - G,\ P_3,\ \lambda > 0\right\rangle :$

   $$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0^2 - x_3^2 - u^2)^{1/2},$$
   $$\omega_3 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} \ ;$$

6. $\left\langle -\frac{1}{\lambda} L_3 - G,\ X_4,\ \lambda > 0\right\rangle :$

$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$
$$\omega_3 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} \ ;$$

7. $\left\langle -\frac{1}{\lambda} L_3 - G - \frac{\alpha}{\lambda} X_3,\ X_4,\ \lambda > 0, \alpha > 0 \right\rangle :$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_3 + \alpha \arctan \frac{x_1}{x_2},$$
$$\omega_3 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} \ .$$

1.4 Classification of functional bases of invariants for three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$

In this section, we present the results of the classification of functional bases of invariants in the space $M(1,3) \times R(u)$ for all three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$.

Taking into account one-dimensional Lie algebras of the type A_1 as well as two-dimensional Lie algebras, we obtain two types of the three-dimensional decomposable Lie algebras: $3A_1$, $A_2 \oplus A_1$. Besides it, there exist 9 types of the real indecomposable Lie algebras $A_{3,1}$, ..., $A_{3,9}$ [63, 66]. Two of them depend on parameters (constitute continuums of the Lie algebras).

In the following, the symbol $A^a_{r,j}$ denotes the jth Lie algebra of dimension r and a is a continuous parameter for the algebra.

It should be indicate that the notation $A^a_{r,j}$ corresponds to those used in the paper by J. Patera et al. [66].

In what follows, for the given specific Lie algebra, we write only nonzero commutation relations [63, 66].

Below, we present the results obtained.

1.4.1 Lie Algebras of the Type $3A_1$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $3A_1$ of the Lie algebra of the group $P(1,4)$ can be formulated as:

Proposition. *The Lie algebra of the group $P(1,4)$ contains 31 three-dimensional nonconjugate subalgebras of the type $3A_1$.*

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\langle G\rangle \oplus \langle X_2\rangle \oplus \langle X_1\rangle$:

   $\omega_1 = x_3, \quad \omega_2 = (x_0^2 - u^2)^{1/2}$;

2. $\langle G\rangle \oplus \langle L_3\rangle \oplus \langle X_3\rangle$:

   $\omega_1 = (x_0^2 - u^2)^{1/2}, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}$;

3. $\langle G + \alpha X_3,\ \alpha > 0\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$:

   $\omega_1 = x_3 - \alpha \ln(x_0 + u), \quad \omega_2 = x_0^2 - u^2$;

4. $\langle L_3\rangle \oplus \langle P_3 + C_3\rangle \oplus \langle X_0 + X_4\rangle$:

   $\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_3^2 + u^2)^{1/2}$;

   It should be noted that next functional bases are invariant with respect to corresponding nonconjugate subalgebras of the type $3A_1$ of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

5. $\langle P_1\rangle \oplus \langle P_2\rangle \oplus \langle X_3\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_0^2 - x_1^2 - x_2^2 - u^2 \ ;$$

6. $\langle P_1\rangle \oplus \langle P_2 - X_2\rangle \oplus \langle X_3\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = \frac{x_0^2 - x_1^2 - u^2}{x_0 + u} - \frac{x_2^2}{x_0 + u + 1} \ ;$$

7. $\langle P_1\rangle \oplus \langle P_2 - \alpha X_2,\ \alpha > 0\rangle \oplus \langle P_3 - \gamma X_3,\ \gamma \neq 0\rangle$:

$$\omega_1 = x_0 + u,$$
$$\omega_2 = 2u + \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} + \frac{x_3^2}{x_0 + u + \gamma} \ ;$$

8. $\langle P_1\rangle \oplus \langle P_2 - \alpha X_2,\ \alpha > 0\rangle \oplus \langle P_3\rangle$:

$$\omega_1 = x_0 + u,$$
$$\omega_2 = 2u + \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} + \frac{x_3^2}{x_0 + u} \ ;$$

9. $\langle P_1\rangle \oplus \langle P_2\rangle \oplus \langle X_4\rangle$:

$$\omega_1 = x_3, \quad \omega_2 = x_0 + u \ ;$$

10. $\langle P_1\rangle \oplus \langle P_2\rangle \oplus \langle P_3\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 \ ;$$

11. $\langle P_1 - \gamma X_3, \gamma > 0\rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta \neq 0\rangle \oplus \langle X_4\rangle$:

$$\omega_1 = x_0 + u,$$
$$\omega_2 = x_3(x_0 + u)^2 - (\gamma x_1 + x_2\delta - x_3)(x_0 + u) - \gamma x_1 \ ;$$

12. $\langle P_1 - \gamma X_3,\ \gamma > 0\rangle \oplus \langle P_2 - X_2\rangle \oplus \langle X_4\rangle$:

$$\omega_1 = x_0 + u,$$
$$\omega_2 = x_3(x_0+u)^2 - (\gamma x_1 - x_3)(x_0+u) - \gamma x_1 \ ;$$

13. $\langle P_1 \rangle \oplus \langle P_2 - X_2 - \delta X_3,\ \delta > 0\rangle \oplus \langle X_4 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_3(x_0+u) - x_2\delta + x_3 \ ;$$

14. $\langle P_1 \rangle \oplus \langle P_2 - X_2 \rangle \oplus \langle X_4 \rangle$:

$$\omega_1 = x_3, \quad \omega_2 = x_0 + u \ ;$$

15. $\langle P_1 - X_3 \rangle \oplus \langle P_2 \rangle \oplus \langle X_4 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_3 - \frac{x_1}{x_0+u} \ ;$$

16. $\langle P_3 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle$:

$$\omega_1 = x_2, \quad \omega_2 = x_0 + u \ ;$$

17. $\langle P_3 \rangle \oplus \langle X_1 \rangle \oplus \langle X_2 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_0^2 - x_3^2 - u^2 \ ;$$

18. $\langle P_3 - X_2 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_2 - \frac{x_3}{x_0+u} \ ;$$

19. $\langle P_3 - 2X_0 \rangle \oplus \langle X_1 \rangle \oplus \langle X_2 \rangle$:

$$\omega_1 = (x_0+u)^2 + 4x_3,$$
$$\omega_2 = x_0 - u + \frac{1}{6}(x_0+u)^3 + x_3(x_0+u) \ ;$$

20. $\langle P_3 - 2X_0 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle$:

$$\omega_1 = x_2, \quad \omega_2 = (x_0 + u)^2 + 4x_3 \ ;$$

21. $\langle L_3 \rangle \oplus \langle X_3 \rangle \oplus \langle X_4 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2} \ ;$$

22. $\langle L_3 \rangle \oplus \langle P_3 \rangle \oplus \langle X_4 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2} \ ;$$

23. $\langle L_3 + \alpha(X_0 + X_4), \ \alpha > 0 \rangle \oplus \langle X_3 \rangle \oplus \langle X_4 \rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_0 + u + \alpha \arctan \frac{x_2}{x_1} \ ;$$

24. $\langle L_3 \rangle \oplus \langle -P_3 + 2X_0 \rangle \oplus \langle 2X_4 \rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0 + u)^2 + 4x_3 \ ;$$

25. $\langle L_3 \rangle \oplus \langle X_4 - X_0 \rangle \oplus \langle X_3 \rangle$:

$$\omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2} \ ;$$

26. $\langle L_3 \rangle \oplus \langle X_0 + X_4 \rangle \oplus \langle X_4 - X_0 \rangle$:

$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2} \ ;$$

27. $\langle L_3 + \alpha(X_0 + X_4), \ \alpha > 0 \rangle \oplus \langle X_3 \rangle \oplus \langle X_4 - X_0 \rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_0 + \alpha \arctan \frac{x_2}{x_1} \ ;$$

28. $\langle L_3 + \alpha X_3, \ \alpha > 0 \rangle \oplus \langle X_0 + X_4 \rangle \oplus \langle X_4 - X_0 \rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_3 + \alpha \arctan \frac{x_1}{x_2} \ .$$

29. $\langle X_0 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle$:

$\omega_1 = x_2, \quad \omega_2 = x_3$;

30. $\langle X_1 \rangle \oplus \langle X_2 \rangle \oplus \langle X_4 - X_0 \rangle$:

    $\omega_1 = x_0, \quad \omega_2 = x_3$;

31. $\langle X_1 \rangle \oplus \langle X_2 \rangle \oplus \langle X_4 \rangle$:

    $\omega_1 = x_3, \quad \omega_2 = x_0 + u$.

1.4.2 Lie Algebras of the Type $A_2 \oplus A_1$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_2 \oplus A_1$ of the Lie algebra of the group $P(1,4)$ can be formulated as:

Proposition. *The Lie algebra of the group* $P(1,4)$ *contains* 10 *three-dimensional nonconjugate subalgebras of the type* $A_2 \oplus A_1$.

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\langle -G,\ P_3 \rangle \oplus \langle X_1 \rangle$:

   $\omega_1 = x_2, \quad \omega_2 = (x_0^2 - x_3^2 - u^2)^{1/2}$;

2. $\langle -G,\ P_3 \rangle \oplus \langle L_3 \rangle$:

   $\omega_1 = (x_0^2 - x_3^2 - u^2)^{1/2}, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}$;

3. $\langle -G - \alpha X_2,\ P_3,\ \alpha > 0 \rangle \oplus \langle X_1 \rangle$:

   $\omega_1 = x_2 - \alpha \ln(x_0 + u), \quad \omega_2 = (x_0^2 - x_3^2 - u^2)^{1/2}$;

4. $\langle -G - \alpha X_2,\ X_4,\ \alpha > 0 \rangle \oplus \langle X_1 \rangle$:

$$\omega_1 = x_3, \quad \omega_2 = x_2 - \alpha \ln(x_0 + u) \ ;$$

5. $\langle -G - \alpha X_3,\ X_4,\ \alpha > 0\rangle \oplus \langle L_3 + \beta X_3,\ \beta > 0\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_3 - \alpha \ln(x_0 + u) + \beta \arctan \frac{x_1}{x_2} \ ;$$

6. $\langle -G - \alpha X_3,\ X_4,\ \alpha > 0\rangle \oplus \langle L_3\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_3 - \alpha \ln(x_0 + u) \ ;$$

7. $\langle -G,\ X_4\rangle \oplus \langle X_1\rangle$:

$$\omega_1 = x_2, \quad \omega_2 = x_3 \ ;$$

8. $\langle -G,\ X_4\rangle \oplus \langle L_3\rangle$:

$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2} \ ;$$

9. $\langle -G,\ X_4\rangle \oplus \langle L_3 + \alpha X_3,\ \alpha > 0\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_3 + \alpha \arctan \frac{x_1}{x_2} \ .$$

10. $\left\langle -\frac{1}{\lambda} L_3 - G,\ 2X_4,\ \lambda > 0 \right\rangle \oplus \langle X_3\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} \ .$$

1.4.3 Lie Algebras of the Type $A_{3,1}$

$[e_2, e_3] = e_1$

The Lie algebras of the type $A_{3,1}$ are nilpotent [66].

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,1}$ of the Lie algebra of the group $P(1,4)$ can be formulated as:

Proposition. *The Lie algebra of the group $P(1,4)$ contains 17 three-dimensional nonconjugate subalgebras of the type $A_{3,1}$.*

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\langle 2\mu X_4,\ P_3,\ X_1 + \mu X_3,\ \mu > 0 \rangle$:

   $\omega_1 = x_2, \quad \omega_2 = x_0 + u$;

2. $\langle 2X_4,\ P_3 - L_3,\ X_3 \rangle$:

   $\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}$;

3. $\langle 2X_4,\ P_3 - X_1,\ X_3 \rangle$:

   $\omega_1 = x_0 + u, \quad \omega_2 = x_2$;

4. $\langle 2\mu X_4,\ P_3 - X_2,\ X_1 + \mu X_3,\ \mu > 0 \rangle$:

   $$\omega_1 = x_0 + u, \quad \omega_2 = x_2 - \frac{x_3 - \mu x_1}{x_0 + u} \ ;$$

5. $\langle -2\alpha X_4,\ L_3 + \alpha X_3,\ P_3,\ \alpha > 0 \rangle$:

   $\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}$;

6. $\langle 4X_4,\ P_1 - X_2 - \gamma X_3,\ P_2 + X_1 - \mu X_2 - \delta X_3,\ \gamma > 0,$
   $\delta \neq 0, \mu > 0 \rangle$:

   $\omega_1 = x_0 + u,$

   $\omega_2 = x_3(x_0 + u)^2 - (\gamma x_1 + x_2 \delta - \mu x_3)(x_0 + u) +$

$+(\delta-\gamma\mu)x_1 - x_2\gamma + x_3$;

7. $\langle 4X_4,\ P_1 - X_2 - \gamma X_3,\ P_2 + X_1 - \mu X_2,\ \gamma > 0, \mu > 0\rangle$:

   $\omega_1 = x_0 + u,$

   $\omega_2 = x_3(x_0+u)^2 - (\gamma x_1 - \mu x_3)(x_0+u) - \gamma\mu x_1 - x_2\gamma + x_3$;

8. $\langle 4X_4,\ P_1 - X_2,\ P_2 + X_1 - \mu X_2 - \delta X_3,\ \delta > 0, \mu \neq 0\rangle$:

   $\omega_1 = x_0 + u,$

   $\omega_2 = x_3(x_0+u)^2 - (x_2\delta - \mu x_3)(x_0+u) +$

   $+\delta x_1 + x_3$;

9. $\langle 4X_4,\ P_1 - X_2,\ P_2 + X_1 - \mu X_2,\ \mu \neq 0\rangle$:

   $\omega_1 = x_0 + u,$

   $\omega_2 = x_3(x_0+u)^2 + \mu x_3(x_0+u) + x_3$;

10. $\langle 4X_4,\ P_1 - X_2,\ P_2 + X_1 - \delta X_3,\ \delta > 0\rangle$:

    $\omega_1 = x_0 + u,$

    $\omega_2 = x_3(x_0+u)^2 - x_2\delta(x_0+u) + \delta x_1 + x_3$;

11. $\langle 4X_4,\ P_1 - X_2,\ P_2 + X_1\rangle$:

    $\omega_1 = x_0 + u, \quad \omega_2 = x_3$;

12. $\langle 4X_4,\ P_1 - X_2 - \beta X_3,\ P_2 + X_1,\ \beta > 0\rangle$:

    $\omega_1 = x_0 + u,$

    $\omega_2 = x_3(x_0+u)^2 - x_1\beta(x_0+u) - \beta x_2 + x_3$;

13. $\langle 2X_4,\ P_3,\ X_3\rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_1, \quad \omega_3 = x_2\ ;$$

As we see, this subalgebra has three invariants instead of two ones. The reason is that this subalgebra has rank equal 2.

14. $\langle 2\mu X_4,\ P_3 - 2X_0,\ X_1 + \mu X_3,\ \mu > 0\rangle$:

$$\omega_1 = x_2, \quad \omega_2 = (x_0 + u)^2 + 4x_3 - 4\mu x_1\ ;$$

15. $\langle 2X_4,\ P_3 - L_3 - 2\alpha X_0,\ X_3,\ \alpha > 0\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = 2\alpha \arctan\frac{x_1}{x_2} - x_0 - u\ ;$$

16. $\langle -2\beta X_4,\ L_3 + \beta X_3,\ P_3 - 2X_0,\ \beta > 0\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \beta \arctan\frac{x_1}{x_2} + \frac{1}{4}(x_0 + u)^2 + x_3\ ;$$

17. $\langle 2X_4,\ P_3 - 2X_0,\ X_3\rangle$:

$$\omega_1 = x_1, \quad \omega_2 = x_2\ .$$

It should be noted that all above written functional bases are invariant with respect to corresponding nonconjugate subalgebras of the type $A_{3,1}$ of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

1.4.4 Lie Algebras of the Type $A_{3,2}$

$$[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$$

The Lie algebras of the type $A_{3,2}$ are solvable [63, 66].

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,2}$ of the Lie algebra of the group $P(1,4)$ can be formulated as:

Proposition. *The Lie algebra of the group $P(1,4)$ contains three three-dimensional nonconjugate subalgebras of the type $A_{3,2}$.*

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\langle 2\beta X_4,\ P_3,\ G + \alpha X_1 + \beta X_3,\ \alpha > 0, \beta > 0\rangle$:

   $\omega_1 = x_2, \quad \omega_2 = x_1 - \alpha \ln(x_0 + u)$;

2. $\left\langle 2\alpha X_4,\ \lambda P_3,\ \frac{1}{\lambda} L_3 + G + \frac{\alpha}{\lambda} X_3,\ \lambda > 0, \alpha > 0 \right\rangle$:

   $\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2}$;

3. $\langle 2\alpha X_4,\ P_3,\ G + \alpha X_3,\ \alpha > 0\rangle$:

   $\omega_1 = x_1, \quad \omega_2 = x_2$.

1.4.5 Lie Algebras of the Type $A_{3,3}$

$[e_1, e_3] = e_1, [e_2, e_3] = e_2$

The Lie algebras of the type $A_{3,3}$ are solvable [63, 66].

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,3}$ of the Lie algebra of the group $P(1,4)$ can be formulated as:

Proposition. *The Lie algebra of the group $P(1,4)$ contains five three-dimensional nonconjugate subalgebras of the type $A_{3,3}$.*

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\langle P_1,\ P_2,\ G\rangle$:

   $$\omega_1 = x_3, \quad \omega_2 = (x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2}\ ;$$

2. $\langle P_1,\ P_2,\ G + \alpha X_3,\ \alpha > 0\rangle$:

   $$\omega_1 = x_3 - \alpha \ln(x_0 + u), \quad \omega_2 = x_0^2 - x_1^2 - x_2^2 - u^2\ ;$$

3. $\left\langle P_3,\ X_4,\ \frac{1}{\lambda}L_3 + G,\ \lambda > 0\right\rangle$:

   $$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2}\ ;$$

4. $\langle P_3,\ X_4,\ G + \alpha X_1,\ \alpha > 0\rangle$:

   $$\omega_1 = x_2, \quad \omega_2 = x_1 - \alpha \ln(x_0 + u)\ ;$$

5. $\langle P_3,\ X_4,\ G\rangle$:

   $$\omega_1 = x_1, \quad \omega_2 = x_2\ .$$

1.4.6 Lie Algebras of the Type $A_{3,4}$

$[e_1, e_3] = e_1, [e_2, e_3] = -e_2$

The Lie algebras of the type $A_{3,4}$ are solvable [63, 66].

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,4}$ of the Lie algebra of the group $P(1, 4)$ can be formulated as:

Proposition. *The Lie algebra of the group $P(1, 4)$ contains four three-dimensional nonconjugate subalgebras of the type $A_{3,4}$.*

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\langle X_0,\ X_4,\ -G-\alpha X_1,\ \alpha>0\rangle$:

   $\omega_1 = x_2, \quad \omega_2 = x_3$;

2. $\left\langle X_0,\ X_4,\ -\frac{1}{\lambda}L_3 - G - \frac{\alpha}{\lambda}X_3,\ \lambda>0, \alpha>0\right\rangle$:

   $\omega_1 = (x_1^2+x_2^2)^{1/2}, \quad \omega_2 = x_3 + \alpha \arctan\frac{x_1}{x_2}$;

3. $\left\langle X_0,\ -X_4,\ -\frac{1}{\lambda}L_3 - G,\ \lambda>0\right\rangle$:

   $\omega_1 = x_3, \quad \omega_2 = (x_1^2+x_2^2)^{1/2}$.

4. $\langle X_4,\ X_0,\ G\rangle$:

   $\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3$.

As we see, this subalgebra has three invariants instead of two ones. The reason is that this subalgebra has rank equal 2.

1.4.7 Lie Algebras of the Type $A^a_{3,5}$

$[e_1,e_3]=e_1, [e_2,e_3]=ae_2, (0<|a|<1)$

The Lie algebra of the group $P(1,4)$ contains no of this type subalgebras.

1.4.8 Lie Algebras of the Type $A_{3,6}$

$[e_1,e_3]=-e_2, [e_2,e_3]=e_1$

Lie algebras of the type $A_{3,6}$ are solvable [63, 66].

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,6}$ of the Lie algebra of the group $P(1,4)$ can be formulated as:

Proposition. *The Lie algebra of the group $P(1,4)$ contains 18 three-dimensional nonconjugate subalgebras of the type $A_{3,6}$.*

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\left\langle X_1, -X_2, -L_3 - \frac{1}{2}(P_3 + C_3) - \alpha(X_0 + X_4), \alpha > 0 \right\rangle$:

   $\omega_1 = (x_3^2 + u^2)^{1/2}, \quad \omega_2 = \alpha \arctan \frac{x_3}{u} - x_0$;

2. $\left\langle X_1,\ X_2,\ L_3 + \frac{\lambda}{2}(P_3 + C_3) + \alpha(X_0 + X_4), 0 < \lambda < 1,\right.$ $\left.\alpha > 0\right\rangle$:

   $\omega_1 = (x_3^2 + u^2)^{1/2}, \quad \omega_2 = \alpha \arctan \frac{x_3}{u} - \lambda x_0$;

3. $\langle -X_1,\ X_2,\ -L_3 - \lambda G,\ \lambda > 0\rangle$:

   $\omega_1 = x_3, \quad \omega_2 = (x_0^2 - u^2)^{1/2}$;

4. $\langle X_1,\ X_2,\ L_3 + \lambda G + \alpha X_3,\ \lambda > 0, \alpha > 0\rangle$:

   $\omega_1 = (x_0^2 - u^2)^{1/2}, \quad \omega_2 = \lambda x_3 - \alpha \ln(x_0 + u)$;

5. $\left\langle X_1,\ X_2,\ L_3 + \frac{1}{2}(P_3 + C_3) \right\rangle$:

   $\omega_1 = x_0, \quad \omega_2 = (x_3^2 + u^2)^{1/2}$;

6. $\left\langle -X_1,\ X_2,\ -L_3 - \frac{\lambda}{2}(P_3 + C_3),\ 0 < \lambda < 1 \right\rangle$:

$$\omega_1 = x_0, \quad \omega_2 = (x_3^2 + u^2)^{1/2} \ ;$$

7. $\left\langle -X_3,\ X_4 - X_0,\ -\frac{1}{\lambda}L_3 - \frac{1}{2}(P_3 + C_3),\ 0 < \lambda < 1 \right\rangle$:

$$\omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2} \ ;$$

8. $\left\langle X_3,\ X_4 - X_0,\ \frac{1}{\lambda}L_3 + \frac{1}{2}(P_3 + C_3) + \frac{\alpha}{\lambda}(X_0 + X_4),\right.$ $\left. 0 < \lambda < 1, \alpha > 0 \right\rangle$:

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \alpha \arctan \frac{x_1}{x_2} - x_0 \ .$$

It should be noted that next functional bases are invariant with respect to corresponding nonconjugate subalgebras of the type $A_{3,6}$ of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

9. $\langle P_1 - X_1,\ P_2 - X_2,\ -P_3 + L_3 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = \frac{x_1^2 + x_2^2}{x_0 + u + 1} + \frac{x_3^2}{x_0 + u} + 2u \ ;$$

10. $\langle P_1,\ -P_2,\ -L_3 - \alpha X_3,\ \alpha > 0 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_0^2 - x_1^2 - x_2^2 - u^2 \ ;$$

11. $\langle P_1,\ P_2,\ -P_3 + L_3 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 \ ;$$

12. $\langle P_1,\ P_2,\ L_3 + 2X_4 \rangle$:

$$\omega_1 = x_3, \quad \omega_2 = x_0 + u \ ;$$

13. $\langle P_1,\ P_2,\ L_3 \rangle$:

$$\omega_1 = x_3, \quad \omega_2 = x_0 + u, \quad \omega_3 = x_0^2 - x_1^2 - x_2^2 - u^2 \ .$$

As we see, this subalgebra has three invariants instead of two ones. The reason is that this subalgebra has rank equal 2.

14. $\langle X_1,\ -X_2,\ P_3 - L_3 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_0^2 - x_3^2 - u^2 \ ;$$

15. $\langle X_1,\ -X_2,\ -L_3 - \alpha X_3,\ \alpha > 0 \rangle$:

$$\omega_1 = x_0, \quad \omega_2 = u \ ;$$

16. $\langle X_1,\ -X_2,\ -L_3 - 2X_4 \rangle$:

$$\omega_1 = x_0 + u, \quad \omega_2 = x_3 \ ;$$

17. $\langle X_1,\ -X_2,\ P_3 - L_3 - 2\alpha X_0,\ \alpha > 0 \rangle$:

$$\omega_1 = (x_0 + u)^2 + 4x_3\alpha,$$
$$\omega_2 = (x_0 + u)^3 + 6\alpha x_3(x_0 + u) + 6\alpha^2(x_0 - u) \ ;$$

18. $\langle X_1,\ X_2,\ L_3 \rangle$:

$$\omega_1 = x_0, \quad \omega_2 = x_3, \quad \omega_3 = u \ .$$

As we see, this subalgebra has three invariants instead of two ones. The reason is that this subalgebra has rank equal 2.

1.4.9 Lie Algebras of the Type $A^a_{3,7}$

$[e_1, e_3] = ae_1 - e_2, [e_2, e_3] = e_1 + ae_2, (a > 0)$

Lie algebras of the type $A^a_{3,7}$ are solvable [63, 66].

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A^a_{3,7}$ of the Lie algebra of the group $P(1, 4)$ can be formulated as:

Proposition. *The Lie algebra of the group $P(1, 4)$ contains two three-dimensional nonconjugate subalgebras of the type $A^a_{3,7}$.*

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\langle P_1,\ P_2,\ L_3 + \lambda G,\ \lambda > 0\rangle :$

   $\omega_1 = x_3, \quad \omega_2 = (x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2}$;

2. $\langle P_1,\ P_2,\ L_3 + \lambda G + \alpha X_3,\ \lambda > 0, \alpha > 0\rangle :$

   $\omega_1 = \lambda x_3 - \alpha \ln(x_0 + u), \quad \omega_2 = x_0^2 - x_1^2 - x_2^2 - u^2$.

1.4.10 Lie Algebras of the Type $A_{3,8}$

$[e_1, e_3] = -2e_2, [e_1, e_2] = e_1, [e_2, e_3] = e_3$

Lie algebras of the type $A_{3,8}$ are semisimple [66].

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,8}$ of the Lie algebra of the group $P(1, 4)$ can be formulated as:

Proposition. *The Lie algebra of the group $P(1, 4)$ contains one three-dimensional nonconjugate subalgebra of the type $A_{3,8}$.*

Below, we present bases elements of that subalgebra and a functional basis of invariants corresponding with it:

$\langle P_3,\ G,\ -C_3\rangle :$

$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = (x_0^2 - x_3^2 - u^2)^{1/2}.$

As we see, this subalgebra has three invariants instead of two ones. The reason is that this subalgebra has rank equal 2.

1.4.11 Lie Algebras of the Type $A_{3,9}$

$[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$

The Lie algebras of the type $A_{3,9}$ are semisimple [66].

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,9}$ of the Lie algebra of the group $P(1,4)$ can be formulated as:

Proposition. *The Lie algebra of the group $P(1,4)$ contains two three-dimensional nonconjugate subalgebras of the type $A_{3,9}$.*

Below, we present bases elements of those subalgebras and functional bases of invariants corresponding with them:

1. $\left\langle -\frac{1}{4}(2L_3 + P_3 + C_3),\ \frac{1}{4}(2L_2 + P_2 + C_2),\right.$
   $\left.\frac{1}{4}(2L_1 + P_1 + C_1)\right\rangle :$

   $\omega_1 = x_0,\ \omega_2 = (x_1^2 + x_2^2 + x_3^2 + u^2)^{1/2}\ ;$

2. $\langle -L_3,\ -L_2,\ -L_1\rangle :$

   $\omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \quad \omega_3 = u\ .$

As we see, this subalgebra has three invariants instead of two ones. The reason is that this subalgebra has rank equal 2.

Chapter 2

Classification of anzatzes for the eikonal equation

In this chapter, we present the results of the classification of anzatzes for the eikonal equation in space $M(1,3) \times R(u)$ for all nonconjugate subalgebras of dimensions 1, 2, and 3 of the Lie algebra of the Poincaré group $P(1,4)$.

The results are obtained using structural properties of low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ [44] as well as the results of the classification of functional bases of invariants for those subalgebras (see Chapter 1).

2.1 Classification of ansatzes for one-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$

In this section, we present the results of the classification of ansatzes in the space $M(1,3) \times R(u)$ for all one-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$.

The results of the classification of one-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group* $P(1,4)$ *contains* 20 *one-dimensional nonconjugate subalgebras of the type* A_1.

However, we only have 19 ansatzes in the space $M(1,3) \times R(u)$, which are invariant with respect to the one-dimensional nonconjugate subalgebras of the type A_1.

Below, we present bases elements of those subalgebras and ansatzes corresponding with them.

1. $\langle G \rangle$:

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_1,\ \omega_2 = x_2, \quad \omega_3 = x_3;$$

2. $\langle G + \alpha X_1,\ \alpha > 0 \rangle$:

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_2, \quad \omega_2 = x_3, \quad \omega_3 = (x_0^2 - u^2)^{1/2};$$

3. $\langle L_3 + \lambda G,\ \lambda > 0 \rangle$:

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$
$$\omega_3 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2};$$

4. $\left\langle L_3 + \frac{1}{2}(P_3 + C_3) \right\rangle$:

$$u^2 + x_3^2 = \varphi(\omega_1, \omega_2, \omega_3),$$

$$\omega_1 = x_0,\ \omega_2 = x_1^2 + x_2^2,\ \omega_3 = \arctan\frac{x_1}{x_2} - \arctan\frac{x_3}{u};$$

5. $\left\langle L_3 + \dfrac{\lambda}{2}(P_3 + C_3),\ 0 < \lambda < 1 \right\rangle :$

$$(u^2 + x_3^2)^{1/2} = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$
$$\omega_3 = \lambda \arctan\frac{x_1}{x_2} - \arctan\frac{x_3}{u};$$

6. $\langle L_3 + \lambda G + \alpha X_3,\ \alpha > 0, \lambda > 0 \rangle :$

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \alpha \ln(x_0 + u) - \lambda x_3,$$
$$\omega_3 = x_3 + \alpha \arctan\frac{x_1}{x_2};$$

7. $\left\langle L_3 + \dfrac{\lambda}{2}(P_3 + C_3) + \alpha(X_0 + X_4),\ \alpha > 0, 0 < \lambda < 1 \right\rangle :$

$$u^2 + x_3^2 = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_1^2 + x_2^2, \quad \omega_2 = x_0 - \alpha \arctan\frac{x_1}{x_2},$$
$$\omega_3 = \lambda x_0 - \alpha \arctan\frac{x_3}{u};$$

8. $\left\langle L_3 + \dfrac{1}{2}(P_3 + C_3) + \alpha(X_0 + X_4),\ \alpha > 0 \right\rangle :$

$$u^2 + x_3^2 = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_1^2 + x_2^2, \quad \omega_2 = x_0 - \alpha \arctan\frac{x_1}{x_2},$$
$$\omega_3 = x_0 - \alpha \arctan\frac{x_3}{u};$$

It should be noted that the next results are obtained with the help of nonconjugate subalgebras of the type A_1 of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

9. $\langle L_3 \rangle$:

$$u = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_0, \quad \omega_2 = x_3, \quad \omega_3 = (x_1^2 + x_2^2)^{1/2};$$

10. $\langle L_3 - P_3 \rangle$:

$$\arctan \frac{x_1}{x_2} + \frac{x_3}{x_0 + u} = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_0 + u, \ \omega_2 = (x_1^2 + x_2^2)^{1/2}, \ \omega_3 = (x_0^2 - x_3^2 - u^2)^{1/2};$$

11. $\langle L_3 + \alpha(X_0 + X_4), \ \alpha > 0 \rangle$:

$$u = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_3, \ \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = x_0 - \alpha \arctan \frac{x_1}{x_2};$$

12. $\langle L_3 + \alpha X_3, \ \alpha > 0 \rangle$:

$$u = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_0, \quad \omega_2 = x_1^2 + x_2^2, \quad \omega_3 = x_3 + \alpha \arctan \frac{x_1}{x_2};$$

13. $\langle L_3 + 2X_4 \rangle$:

$$x_0 - u + 2 \arctan \frac{x_2}{x_1} = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = x_3;$$

14. $\langle L_3 - P_3 + 2\alpha X_0,\ \alpha > 0\rangle :$

$$(x_0+u)^2 + 4x_3\alpha = \varphi(\omega_1,\omega_2,\omega_3),$$
$$\omega_1 = (x_1^2+x_2^2)^{1/2}, \quad \omega_2 = 2\alpha \arctan\frac{x_1}{x_2} - x_0 - u,$$
$$\omega_3 = 2(x_0+u)^3 + 12\alpha^2(x_0-u) + 12\alpha x_3(x_0+u);$$

15. $\langle P_3\rangle :$

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1,\omega_2,\omega_3),$$
$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_0 + u;$$

16. $\langle P_3 - 2X_0\rangle :$

$$x_0 - u + \frac{1}{6}(x_0+u)^3 + x_3(x_0+u) = \varphi(\omega_1,\omega_2,\omega_3),$$
$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = (x_0+u)^2 + 4x_3;$$

17. $\langle P_3 - X_1\rangle :$

$$x_1(x_0+u) - x_3 = \varphi(\omega_1,\omega_2,\omega_3),$$
$$\omega_1 = x_2, \quad \omega_2 = x_0 + u, \quad \omega_3 = x_3^2 + 2u(x_0+u);$$

18. $\langle X_4\rangle :$

$$x_3 = \varphi(\omega_1,\omega_2,\omega_3),$$
$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_0 + u;$$

19. $\langle X_0 + X_4\rangle :$

$$u = \varphi(\omega_1,\omega_2,\omega_3),$$
$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3.$$

From the invariants of the remaining one nonconjugate subalgebra it is impossible to construct an ansatz, which reduces the eikonal equation. Let's present bases elements of that subalgebra as well as a functional basis of invariants corresponding with it.

$\langle X_4 - X_0 \rangle :$

$\omega_1 = x_0, \quad \omega_2 = x_1, \quad \omega_3 = x_2, \quad \omega_4 = x_3.$

2.2 Classification of ansatzes for two-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$

In this section, we present the results of the classification of ansatzes in the space $M(1,3) \times R(u)$ for all two-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$.

2.2.1 Lie Algebras of the Type $2A_1$

The results of the classification of two-dimensional nonconjugate subalgebras of the type $2A_1$ of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group* $P(1,4)$ *contains* 42 *two-dimensional nonconjugate subalgebras of the type* $2A_1$.

However, we only have 37 ansatzes, which are invariant with respect to the two-dimensional nonconjugate subalgebras of the type $2A_1$.

Below, we present bases elements of those subalgebras and ansatzes corresponding with them.

1. $\langle G\rangle \oplus \langle L_3\rangle$:

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}.$$

2. $\langle G + \alpha X_3,\ \alpha > 0\rangle \oplus \langle L_3\rangle$:

$$x_3 - \alpha \ln(x_0 + u) = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0^2 - u^2)^{1/2}.$$

3. $\langle G\rangle \oplus \langle L_3 + \alpha X_3,\ \alpha > 0\rangle$:

$$x_3 + \alpha \arctan\frac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0^2 - u^2)^{1/2}.$$

4. $\langle G + \alpha X_3,\ \alpha > 0\rangle \oplus \langle L_3 + \beta X_3,\ \beta > 0\rangle$:

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_3 - \alpha \ln(x_0 + u) - \beta \arctan\frac{x_2}{x_1}, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}.$$

5. $\langle L_3 + \lambda G,\ \lambda > 0\rangle \oplus \langle X_3\rangle$:

$$(x_1^2 + x_2^2)^{1/2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_0^2 - u^2)^{1/2}, \quad \omega_2 = \ln(x_0 + u) + \lambda \arctan\frac{x_1}{x_2};$$

6. $\langle G\rangle \oplus \langle X_1\rangle$:

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_2, \quad \omega_2 = x_3;$$

7. $\langle G + \alpha X_2,\ \alpha > 0\rangle \oplus \langle X_1\rangle :$

$$x_3 = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_0^2 - u^2)^{1/2}, \quad \omega_2 = x_2 - \alpha \ln(x_0 + u);$$

8. $\langle L_3\rangle \oplus \langle P_3 + C_3\rangle :$

$$(u^2 + x_3^2)^{1/2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2};$$

9. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0\rangle \oplus \langle P_3 + C_3\rangle :$

$$x_0 - \alpha \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (u^2 + x_3^2)^{1/2};$$

10. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0\rangle \oplus \langle P_3 + C_3 + 2\beta(X_0 + X_4),$
$\beta > 0\rangle :$

$$\alpha \arctan \frac{x_1}{x_2} + \beta \arctan \frac{x_3}{u} - x_0 = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (u^2 + x_3^2)^{1/2};$$

11. $\left\langle L_3 + \frac{1}{2}(P_3 + C_3)\right\rangle \oplus \langle X_0 + X_4\rangle :$

$$\arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (u^2 + x_3^2)^{1/2};$$

12. $\left\langle L_3 + \frac{\lambda}{2}(P_3 + C_3),\ 0 < \lambda < 1\right\rangle \oplus \langle X_0 + X_4\rangle :$

$$\lambda \arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u} = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (u^2 + x_3^2)^{1/2}.$$

It should be noted that the next results are obtained with the help of nonconjugate subalgebras of the type $2A_1$ of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

13. $\langle L_3 \rangle \oplus \langle P_3 \rangle :$

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2};$$

14. $\langle L_3 + 2X_4 \rangle \oplus \langle P_3 \rangle :$

$$\frac{x_3^2}{x_0 + u} + 2 \arctan \frac{x_1}{x_2} + 2u = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \ \omega_2 = x_0 + u;$$

15. $\langle L_3 \rangle \oplus \langle P_3 - 2X_0 \rangle :$

$$x_0 - u + \frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0 + u)^2 + 4x_3;$$

16. $\langle L_3 + 2X_4 \rangle \oplus \langle P_3 - 2\beta X_0, \ \beta > 0 \rangle :$

$$(x_0 + u)^2 + 4\beta x_3 = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2},$$

$$\omega_2 = 4\beta \arctan \frac{x_1}{x_2} - 2\beta(x_0 - u) - \frac{1}{3\beta}(x_0 + u)^3 -$$

$$-2x_3(x_0 + u);$$

17. $\langle L_3 - P_3 \rangle \oplus \langle X_4 \rangle$:

$$x_0 + u = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \arctan \frac{x_1}{x_2} + \frac{x_3}{x_0 + u};$$

18. $\langle L_3 - P_3 + 2\alpha X_0,\ \alpha \neq 0 \rangle \oplus \langle X_4 \rangle$:

$$x_0 + u - 2\alpha \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0 + u)^2 + 4\alpha x_3;$$

19. $\langle L_3 \rangle \oplus \langle X_0 + X_4 \rangle$:

$$u = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_3;$$

20. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0 \rangle \oplus \langle X_4 \rangle$:

$$x_0 + u - \alpha \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2};$$

21. $\langle L_3 + \alpha X_3,\ \alpha > 0 \rangle \oplus \langle X_0 + X_4 \rangle$:

$$x_3 + \alpha \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2};$$

22. $\langle L_3 + 2X_4 \rangle \oplus \langle X_3 \rangle$:

$$x_0 - u + 2 \arctan \frac{x_2}{x_1} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2};$$

23. $\langle L_3 \rangle \oplus \langle X_4 \rangle$:

$$(x_1^2 + x_2^2)^{1/2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_0 + u, \quad \omega_2 = x_3;$$

24. $\langle L_3 + \alpha X_3,\ \alpha > 0 \rangle \oplus \langle X_4 \rangle$:

$$x_3 + \alpha \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2};$$

25. $\langle P_1 \rangle \oplus \langle P_2 \rangle$:

$$x_0 + u = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_3, \quad \omega_2 = (x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2};$$

26. $\langle P_1 - X_3 \rangle \oplus \langle P_2 \rangle$:

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_0 + u, \quad \omega_2 = x_3 - \frac{x_1}{x_0 + u};$$

27. $\langle P_1 \rangle \oplus \langle P_2 - X_2 - \beta X_3,\ \beta > 0 \rangle$:

$$\frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + 1} + 2u = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_0 + u,\ \omega_2 = x_3 - \beta \frac{x_2}{x_0 + u + 1};$$

28. $\langle P_1 \rangle \oplus \langle P_2 - X_2 \rangle$:

$$\frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + 1} + 2u = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_0 + u,\ \omega_2 = x_3;$$

29. $\langle P_1 - X_3\rangle \oplus \langle P_2 - \gamma X_2,\ \gamma > 0\rangle$:

$$\frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u+\gamma} + 2u = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_0 + u,\ \omega_2 = \frac{x_1}{x_0+u} - x_3;$$

30. $\langle P_1 - X_3\rangle \oplus \langle P_2 - \gamma X_2 - \beta X_3,\ \beta > 0, \gamma > 0\rangle$:

$$\frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u+\gamma} + 2u = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_0 + u, \quad \omega_2 = \frac{\beta x_2}{x_0+u+\gamma} + \frac{x_1}{x_0+u} - x_3;$$

31. $\langle P_3\rangle \oplus \langle X_1\rangle$:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_2, \quad \omega_2 = x_0 + u;$$

32. $\langle P_3 - 2X_0\rangle \oplus \langle X_1\rangle$:

$$x_0 - u + \frac{1}{6}(x_0+u)^3 + x_3(x_0+u) = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_2,\ \omega_2 = (x_0+u)^2 + 4x_3;$$

33. $\langle P_3 - X_2\rangle \oplus \langle X_1\rangle$:

$$x_2 - \frac{x_3}{x_0+u} = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_0^2 - x_3^2 - u^2)^{1/2};$$

34. $\langle P_3\rangle \oplus \langle X_4\rangle$:

$$x_1 = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_2, \quad \omega_2 = x_0 + u;$$

35. $\langle P_3 - 2X_0 \rangle \oplus \langle X_4 \rangle$:

$$(x_0 + u)^2 + 4x_3 = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_1, \quad \omega_2 = x_2;$$

36. $\langle P_3 - X_1 \rangle \oplus \langle X_4 \rangle$:

$$x_1 - \frac{x_3}{x_0 + u} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_2, \quad \omega_2 = x_0 + u;$$

37. $\langle X_1 \rangle \oplus \langle X_4 \rangle$:

$$x_3 = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_0 + u, \quad \omega_2 = x_2.$$

From the invariants of the remaining five nonconjugate subalgebras it is impossible to construct ansatzes, which reduce the eikonal equation. Let's present bases elements of those subalgebras as well as functional bases of invariants corresponding with them.

1. $\langle L_3 \rangle \oplus \langle X_4 - X_0 \rangle$:
$$\omega_1 = x_0, \quad \omega_2 = x_3, \quad \omega_3 = (x_1^2 + x_2^2)^{1/2} .$$

2. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0 \rangle \oplus \langle X_4 - X_0 \rangle$:
$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = x_0 - \alpha \arctan \frac{x_1}{x_2} .$$

3. $\langle L_3 + \alpha X_3,\ \alpha > 0 \rangle \oplus \langle X_4 - X_0 \rangle$:
$$\omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = x_3 + \alpha \arctan \frac{x_1}{x_2} .$$

4. $\langle X_0 + X_4 \rangle \oplus \langle X_4 - X_0 \rangle$:

$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3\ .$$

5. $\langle X_1 \rangle \oplus \langle X_4 - X_0 \rangle$:

$$\omega_1 = x_0, \quad \omega_2 = x_2, \quad \omega_3 = x_3\ .$$

2.2.2 Lie Algebras of the Type A_2

The results of the classification of two-dimensional nonconjugate subalgebras of the type A_2 of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group $P(1,4)$ contains seven two-dimensional nonconjugate subalgebras of the type A_2.*

However, we only have six ansatzes, which are invariant with respect to two-dimensional nonconjugate subalgebras of the type A_2.

Below, we present bases elements of those subalgebras and ansatzes corresponding with them.

1. $\langle -G,\ P_3 \rangle$:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_1, \quad \omega_2 = x_2;$$

2. $\left\langle -G - \dfrac{1}{\lambda} L_3,\ P_3,\ \lambda > 0 \right\rangle$:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2};$$

3. $\left\langle -G - \frac{1}{\lambda}L_3,\ X_4,\ \lambda > 0 \right\rangle$:

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2};$$

4. $\langle -G - \alpha X_1,\ X_4,\ \alpha > 0 \rangle$:

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_2,\ \omega_2 = x_3;$$

5. $\left\langle -\frac{1}{\lambda}(L_3 + \lambda G + \alpha X_3),\ X_4,\ \alpha > 0, \lambda > 0 \right\rangle$:

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2},\ \omega_2 = x_3 + \alpha \arctan \frac{x_1}{x_2};$$

6. $\langle -G - \alpha X_1,\ P_3,\ \alpha > 0 \rangle$:

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_2, \quad \omega_2 = (x_0^2 - x_3^2 - u^2)^{1/2}.$$

From the invariants of the remaining one nonconjugate subalgebra it is impossible to construct ansatz, which reduces the eikonal equation. Let's present bases elements of this subalgebra and, corresponding to its, functional basis of invariants.

$\langle -G,\ X_4 \rangle$:

$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3\ .$$

2.3 Classification of anzatzes for the three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$

In this section, we present the results of the classification of ansatzes in the space $M(1,3) \times R(u)$ for all three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$.

2.3.1 Lie Algebras of the Type $3A_1$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $3A_1$ of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group* $P(1,4)$ *contains 31 three-dimensional nonconjugate subalgebras of the type* $3A_1$.

However, we only have 25 ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $3A_1$.

Below, we present bases elements of those subalgebras and ansatzes corresponding with them.

1. $\langle P_1 - \gamma X_3, \gamma > 0 \rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta \neq 0 \rangle \oplus \langle X_4 \rangle :$

   $x_3(x_0+u)^2 - (\gamma x_1 + x_2\delta - x_3)(x_0+u) - \gamma x_1 = \varphi(\omega),$

   $\omega = x_0 + u;$

2. $\langle P_1 - \gamma X_3, \gamma > 0 \rangle \oplus \langle P_2 - X_2 \rangle \oplus \langle X_4 \rangle :$

   $x_3(x_0+u)^2 - (\gamma x_1 - x_3)(x_0+u) - \gamma x_1 = \varphi(\omega),$

   $\omega = x_0 + u;$

3. $\langle P_1 \rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta > 0 \rangle \oplus \langle X_4 \rangle$:

$$x_3(x_0 + u) - x_2\delta + x_3 = \varphi(\omega), \quad \omega = x_0 + u;$$

4. $\langle P_1 - X_3 \rangle \oplus \langle P_2 \rangle \oplus \langle X_4 \rangle$:

$$x_3 - \frac{x_1}{x_0 + u} = \varphi(\omega), \quad \omega = x_0 + u;$$

5. $\langle P_3 - X_2 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle$:

$$x_2 - \frac{x_3}{x_0 + u} = \varphi(\omega), \quad \omega = x_0 + u;$$

6. $\langle P_1 \rangle \oplus \langle P_2 \rangle \oplus \langle X_3 \rangle$:

$$x_0^2 - x_1^2 - x_2^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u;$$

7. $\langle P_3 \rangle \oplus \langle X_1 \rangle \oplus \langle X_2 \rangle$:

$$x_0^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u;$$

8. $\langle P_1 \rangle \oplus \langle P_2 \rangle \oplus \langle P_3 \rangle$:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u;$$

9. $\langle P_1 \rangle \oplus \langle P_2 - X_2 \rangle \oplus \langle X_3 \rangle$:

$$\frac{x_0^2 - x_1^2 - u^2}{x_0 + u} - \frac{x_2^2}{x_0 + u + 1} = \varphi(\omega), \quad \omega = x_0 + u;$$

10. $\langle P_1 \rangle \oplus \langle P_2 - \alpha X_2, \alpha > 0 \rangle \oplus \langle P_3 - \gamma X_3, \gamma \neq 0 \rangle$:

$$2u + \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} + \frac{x_3^2}{x_0 + u + \gamma} = \varphi(\omega),$$
$$\omega = x_0 + u;$$

11. $\langle P_1\rangle \oplus \langle P_2 - \alpha X_2, \alpha > 0\rangle \oplus \langle P_3\rangle$:

$$2u + \frac{x_1^2 + x_3^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} = \varphi(\omega), \quad \omega = x_0 + u;$$

12. $\langle X_1\rangle \oplus \langle X_2\rangle \oplus \langle X_4\rangle$:

$$x_0 + u = \varphi(\omega), \quad \omega = x_3;$$

13. $\langle P_1\rangle \oplus \langle P_2 - X_2\rangle \oplus \langle X_4\rangle$:

$$x_0 + u = \varphi(\omega), \quad \omega = x_3;$$

14. $\langle L_3\rangle \oplus \langle X_3\rangle \oplus \langle X_4\rangle$:

$$x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2};$$

15. $\langle P_3\rangle \oplus \langle X_1\rangle \oplus \langle X_4\rangle$:

$$x_0 + u = \varphi(\omega), \quad \omega = x_2;$$

16. $\langle L_3\rangle \oplus \langle P_3\rangle \oplus \langle X_4\rangle$:

$$x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2};$$

17. $\langle P_1\rangle \oplus \langle P_2\rangle \oplus \langle X_4\rangle$:

$$x_0 + u = \varphi(\omega), \quad \omega = x_3;$$

18. $\langle G\rangle \oplus \langle X_2\rangle \oplus \langle X_1\rangle$:

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3;$$

19. $\langle G\rangle \oplus \langle L_3\rangle \oplus \langle X_3\rangle$:

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2};$$

20. $\langle P_3 - 2X_0\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$:

$$\frac{1}{6}(x_0+u)^3 + x_3(x_0+u) + x_0 - u = \varphi(\omega),$$
$$\omega = (x_0+u)^2 + 4x_3;$$

21. $\langle G + \alpha X_3, \alpha > 0\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$:

$$x_3 - \alpha \ln(x_0+u) = \varphi(\omega), \quad \omega = x_0^2 - u^2;$$

22. $\langle L_3\rangle \oplus \langle P_3 + C_3\rangle \oplus \langle X_0 + X_4\rangle$:

$$(x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2};$$

23. $\langle L_3 + \alpha(X_0 + X_4), \alpha > 0\rangle \oplus \langle X_3\rangle \oplus \langle X_4\rangle$:

$$x_0 + u + \alpha \arctan \frac{x_2}{x_1} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2};$$

24. $\langle P_3 - 2X_0\rangle \oplus \langle X_1\rangle \oplus \langle X_4\rangle$:

$$(x_0+u)^2 + 4x_3 = \varphi(\omega), \quad \omega = x_2;$$

25. $\langle L_3\rangle \oplus \langle -P_3 + 2X_0\rangle \oplus \langle 2X_4\rangle$:

$$(x_0+u)^2 + 4x_3 = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

From the invariants of the remaining six nonconjugate subalgebras it is impossible to construct the ansatzes, which reduce the eikonal equation.

Below, we present bases elements of those subalgebras and their invariants.

1. $\langle L_3\rangle \oplus \langle X_0 + X_4\rangle \oplus \langle X_4 - X_0\rangle$: x_3, $(x_1^2 + x_2^2)^{1/2}$;

2. $\langle L_3 + \alpha(X_0 + X_4), \alpha > 0\rangle \oplus \langle X_3\rangle \oplus \langle X_4 - X_0\rangle$:

   $(x_1^2 + x_2^2)^{1/2},\ x_0 + \alpha \arctan \dfrac{x_2}{x_1};$

3. $\langle L_3 + \alpha X_3, \alpha > 0\rangle \oplus \langle X_0 + X_4\rangle \oplus \langle X_4 - X_0\rangle$:

   $(x_1^2 + x_2^2)^{1/2},\ x_3 + \alpha \arctan \dfrac{x_1}{x_2};$

4. $\langle L_3\rangle \oplus \langle X_4 - X_0\rangle \oplus \langle X_3\rangle$: $x_0,\ (x_1^2 + x_2^2)^{1/2};$

5. $\langle X_0\rangle \oplus \langle X_1\rangle \oplus \langle X_4\rangle$: $x_2,\ x_3;$

6. $\langle X_1\rangle \oplus \langle X_2\rangle \oplus \langle X_4 - X_0\rangle$: $x_0,\ x_3.$

2.3.2 Lie Algebras of the Type $A_2 \oplus A_1$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_2 \oplus A_1$ of the Lie algebra of the group $P(1, 4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group $P(1, 4)$ contains 10 three-dimensional nonconjugate subalgebras of the type $A_2 \oplus A_1$.*

However, we only have seven ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_2 \oplus A_1$.

Below, we present bases elements of those subalgebras and ansatzes corresponding with them.

1. $\langle -G,\ P_3\rangle \oplus \langle X_1\rangle$:

   $(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_2;$

2. $\langle -G,\ P_3\rangle \oplus \langle L_3\rangle$:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2};$$

3. $\langle -(G + \alpha X_2),\ P_3, \alpha > 0\rangle \oplus \langle X_1\rangle$:

$$x_2 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = (x_0^2 - x_3^2 - u^2)^{1/2};$$

4. $\left\langle -\frac{1}{\lambda} L_3 - G,\ 2X_4, \lambda > 0 \right\rangle \oplus \langle X_3\rangle$:

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2};$$

5. $\langle -(G + \alpha X_2),\ X_4, \alpha > 0\rangle \oplus \langle X_1\rangle$:

$$x_2 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_3;$$

6. $\langle -(G + \alpha X_3),\ X_4, \alpha > 0\rangle \oplus \langle L_3 + \beta X_3, \beta > 0\rangle$:

$$x_3 - \alpha \ln(x_0 + u) + \beta \arctan \frac{x_1}{x_2} = \varphi(\omega),$$

$$\omega = (x_1^2 + x_2^2)^{1/2};$$

7. $\langle -(G + \alpha X_3),\ X_4, \alpha > 0\rangle \oplus \langle L_3\rangle$:

$$x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

From the invariants of the remaining three nonconjugate subalgebras it is impossible to construct the ansatzes, which reduce the eikonal equation.

Below, we present bases elements of those subalgebras and their invariants.

1. $\langle -G,\ X_4\rangle \oplus \langle X_1\rangle :\ x_2,\ x_3;$

2. $\langle -G,\ X_4\rangle \oplus \langle L_3\rangle :\ x_3,\ (x_1^2 + x_2^2)^{1/2};$

3. $\langle -G,\ X_4\rangle \oplus \langle L_3+\alpha X_3, \alpha>0\rangle$:

$$(x_1^2+x_2^2)^{1/2}, \quad x_3+\alpha \arctan\frac{x_1}{x_2}.$$

2.3.3 Lie Algebras of the Type $A_{3,1}$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,1}$ of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group $P(1,4)$ contains 17 three-dimensional nonconjugate subalgebras of the type $A_{3,1}$.*

However, we only have 16 ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,1}$.

Below, we present bases elements of those subalgebras and ansatzes corresponding with them.

1. $\langle 4X_4,\ P_1-X_2-\gamma X_3,\ P_2+X_1-\mu X_2-\delta X_3, \gamma>0, \delta\neq 0, \mu>0\rangle$:

$$x_3(x_0+u)^2-(\gamma x_1+x_2\delta-\mu x_3)(x_0+u)+(\delta-\gamma\mu)x_1- -x_2\gamma+x_3=\varphi(\omega), \quad \omega=x_0+u;$$

2. $\langle 4X_4,\ P_1-X_2-\gamma X_3,\ P_2+X_1-\mu X_2, \gamma>0, \mu>0\rangle$:

$$x_3(x_0+u)^2-(\gamma x_1-\mu x_3)(x_0+u)-\gamma\mu x_1-x_2\gamma+x_3= =\varphi(\omega), \quad \omega=x_0+u;$$

3. $\langle 4X_4,\ P_1-X_2,\ P_2+X_1-\mu X_2-\delta X_3, \delta>0, \mu\neq 0\rangle$:

$$x_3(x_0+u)^2-(x_2\delta-\mu x_3)(x_0+u)+\delta x_1+x_3=\varphi(\omega), \quad \omega=x_0+u;$$

4. $\langle 4X_4,\ P_1 - X_2,\ P_2 + X_1 - \delta X_3, \delta > 0\rangle :$

$$x_3(x_0+u)^2 - x_2\delta(x_0+u) + \delta x_1 + x_3 = \varphi(\omega), \quad \omega = x_0+u;$$

5. $\langle 4X_4,\ P_1 - X_2 - \beta X_3,\ P_2 + X_1, \beta > 0\rangle :$

$$x_3(x_0+u)^2 - \beta x_1(x_0+u) - \beta x_2 + x_3 = \varphi(\omega), \quad \omega = x_0+u;$$

6. $\langle 4X_4,\ P_1 - X_2,\ P_2 + X_1 - \mu X_2, \mu \neq 0\rangle :$

$$x_3(x_0+u)^2 + \mu x_3(x_0+u) + x_3 = \varphi(\omega), \quad \omega = x_0+u;$$

7. $\langle 2\mu X_4,\ P_3 - X_2,\ X_1 + \mu X_3, \mu > 0\rangle :$

$$x_2 - \frac{x_3 - \mu x_1}{x_0 + u} = \varphi(\omega), \quad \omega = x_0+u;$$

8. $\langle 2\mu X_4,\ P_3,\ X_1 + \mu X_3, \mu > 0\rangle :$

$$x_0 + u = \varphi(\omega), \quad \omega = x_2;$$

9. $\langle 2X_4,\ P_3 - L_3,\ X_3\rangle :$

$$x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2};$$

10. $\langle 2X_4,\ P_3 - X_1,\ X_3\rangle :$

$$x_0 + u = \varphi(\omega), \quad \omega = x_2;$$

11. $\langle -2\alpha X_4,\ L_3 + \alpha X_3,\ P_3, \alpha > 0\rangle :$

$$x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2};$$

12. $\langle 4X_4,\ P_1 - X_2,\ P_2 + X_1\rangle :$

$$x_0 + u = \varphi(\omega), \quad \omega = x_3;$$

13. $\langle 2\mu X_4,\ P_3 - 2X_0,\ X_1 + \mu X_3, \mu > 0\rangle :$

$$(x_0+u)^2 + 4x_3 - 4\mu x_1 = \varphi(\omega), \quad \omega = x_2;$$

14. $\langle 2X_4,\ P_3 - L_3 - 2\alpha X_0,\ X_3, \alpha > 0\rangle$:

$$2\alpha \arctan \frac{x_1}{x_2} - x_0 - u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2};$$

15. $\langle -2\beta X_4,\ L_3 + \beta X_3,\ P_3 - 2X_0, \beta > 0\rangle$:

$$\beta \arctan \frac{x_1}{x_2} + \frac{1}{4}(x_0 + u)^2 + x_3 = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2};$$

16. $\langle 2X_4,\ P_3,\ X_3\rangle$:

$$x_2 = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 + u,\ \omega_2 = x_1;$$

From the invariants of the remaining one nonconjugate subalgebra it is impossible to construct the ansatz, which reduces the eikonal equation.

Below, we present bases elements of the subalgebra and its invariants.

$\langle 2X_4,\ P_3 - 2X_0,\ X_3\rangle$: $x_1,\ x_2$.

2.3.4 Lie Algebras of the Type $A_{3,2}$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,2}$ of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group $P(1,4)$ contains three three-dimensional nonconjugate subalgebras of the type $A_{3,2}$.*

However, we only have two ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,2}$.

Below, we present bases elements of those subalgebras and ansatzes corresponding with them.

1. $\langle 2\beta X_4,\ P_3,\ G + \alpha X_1 + \beta X_3, \alpha > 0, \beta > 0\rangle$:

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_2;$$

2. $\left\langle 2\alpha X_4,\ \lambda P_3,\ \frac{1}{\lambda} L_3 + G + \frac{\alpha}{\lambda} X_3, \alpha > 0, \lambda > 0 \right\rangle$:

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

From the invariants of the remaining one nonconjugate subalgebra it is impossible to construct the ansatz, which reduces the eikonal equation.

Below, we present bases elements of the subalgebra as well as its invariants.

$\langle 2\alpha X_4,\ P_3,\ G + \alpha X_3, \alpha > 0\rangle$: $x_1,\ x_2$.

2.3.5 Lie Algebras of the Type $A_{3,3}$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,3}$ of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group $P(1,4)$ contains five three-dimensional nonconjugate subalgebras of the type $A_{3,3}$.*

However, we only have four ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,3}$.

Below, we present bases elements of those subalgebras and ansatzes corresponding with them.

1. $\langle P_1,\ P_2,\ G\rangle$:

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3;$$

2. $\langle P_1,\ P_2,\ G+\alpha X_3, \alpha>0\rangle$:

$$x_3-\alpha\ln(x_0+u)=\varphi(\omega),\quad \omega=x_0^2-x_1^2-x_2^2-u^2;$$

3. $\left\langle P_3,\ X_4,\ \frac{1}{\lambda}L_3+G, \lambda>0\right\rangle$:

$$\ln(x_0+u)+\lambda\arctan\frac{x_1}{x_2}=\varphi(\omega),\quad \omega=(x_1^2+x_2^2)^{1/2};$$

4. $\langle P_3,\ X_4,\ G+\alpha X_1, \alpha>0\rangle$:

$$x_1-\alpha\ln(x_0+u)=\varphi(\omega),\quad \omega=x_2.$$

From the invariants of the remaining one nonconjugate subalgebra it is impossible to construct the ansatz, which reduces the eikonal equation.

Below, we present bases elements of the subalgebra and its invariants.

$\langle P_3,\ X_4,\ G\rangle$: $x_1,\ x_2$.

2.3.6 Lie Algebras of the Type $A_{3,4}$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,4}$ of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group $P(1,4)$ contains four three-dimensional nonconjugate subalgebras of the type $A_{3,4}$.*

From the invariants of all four nonconjugate subalgebras it is impossible to construct the ansatzes, which reduce the eikonal equation.

Below, we present bases elements of those subalgebras and their invariants.

1. $\langle X_4,\ X_0,\ G\rangle$: $x_1,\ x_2,\ x_3$;

2. $\left\langle X_0,\ -X_4,\ -\frac{1}{\lambda}L_3 - G, \lambda > 0\right\rangle$: $x_3,\ (x_1^2 + x_2^2)^{1/2}$;

3. $\langle X_0,\ X_4,\ -(G + \alpha X_1), \alpha > 0\rangle$: $x_2,\ x_3$;

4. $\left\langle X_0,\ X_4,\ -\frac{L_3}{\lambda} - G - \frac{\alpha}{\lambda}X_3, \alpha > 0, \lambda > 0\right\rangle$:

   $(x_1^2 + x_2^2)^{1/2},\ x_3 + \alpha \arctan \frac{x_1}{x_2}$.

2.3.7 Lie Algebras of the Type $A_{3,5}^a$

The Lie algebra of the group $P(1,4)$ contains no nonconjugate subalgebras of the type $A_{3,5}^a$.

2.3.8 Lie Algebras of the Type $A_{3,6}$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,6}$ of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group $P(1,4)$ contains 18 three-dimensional nonconjugate subalgebras of the type $A_{3,6}$.*

However, we only have 16 ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,6}$.

Below, we present bases elements of those subalgebras and ansatzes corresponding with them.

1. $\langle P_1 - X_1,\ P_2 - X_2,\ -P_3 + L_3\rangle$:

$$\frac{x_1^2 + x_2^2}{x_0 + u + 1} + \frac{x_3^2}{x_0 + u} + 2u = \varphi(\omega), \quad \omega = x_0 + u;$$

2. $\langle P_1,\ -P_2,\ -\left(L_3 + \alpha X_3\right), \alpha > 0\rangle$:

$$x_0^2 - x_1^2 - x_2^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u;$$

3. $\langle X_1,\ -X_2,\ P_3 - L_3\rangle$:

$$x_0^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u;$$

4. $\langle P_1,\ P_2,\ -P_3 + L_3\rangle$:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u;$$

5. $\langle X_1,\ -X_2,\ -\left(L_3 + 2X_4\right)\rangle$:

$$x_0 + u = \varphi(\omega), \quad \omega = x_3;$$

6. $\langle P_1,\ P_2,\ L_3 + 2X_4\rangle$:

$$x_0 + u = \varphi(\omega), \quad \omega = x_3;$$

7. $\left\langle X_1,\ X_2,\ L_3 + \frac{1}{2}\left(P_3 + C_3\right)\right\rangle$:

$$(x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = x_0;$$

8. $\left\langle -X_1,\ X_2,\ -L_3 - \frac{\lambda}{2}(P_3 + C_3), 0 < \lambda < 1\right\rangle$:

$$(x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = x_0;$$

9. $\langle -X_1,\ X_2,\ -(L_3+\lambda G), \lambda>0\rangle$:

$(x_0^2-u^2)^{1/2}=\varphi(\omega), \quad \omega=x_3;$

10. $\langle X_1,\ -X_2,\ -(L_3+\alpha X_3), \alpha>0\rangle$:

$u=\varphi(\omega), \quad \omega=x_0;$

11. $\langle X_1,\ -X_2,\ P_3-L_3-2\alpha X_0, \alpha>0\rangle$:

$(x_0+u)^3+6\alpha x_3(x_0+u)+6\alpha^2(x_0-u)=\varphi(\omega),$
$\omega=(x_0+u)^2+4x_3\alpha;$

12. $\Big\langle X_1,\ -X_2,\ -L_3-\frac{1}{2}(P_3+C_3)-\alpha(X_0+X_4),$
$\alpha>0\rangle$:

$\alpha\arctan\frac{x_3}{u}-x_0=\varphi(\omega), \quad \omega=(x_3^2+u^2)^{1/2};$

13. $\Big\langle X_1,\ X_2,\ L_3+\frac{\lambda}{2}(P_3+C_3)+\alpha(X_0+X_4), \alpha>0,$
$0<\lambda<1\rangle$:

$\alpha\arctan\frac{x_3}{u}-\lambda x_0=\varphi(\omega), \quad \omega=(x_3^2+u^2)^{1/2};$

14. $\langle X_1,\ X_2,\ L_3+\lambda G+\alpha X_3, \alpha>0, \lambda>0\rangle$:

$\lambda x_3-\alpha\ln(x_0+u)=\varphi(\omega), \quad \omega=(x_0^2-u^2)^{1/2};$

15. $\langle X_1,\ X_2,\ L_3\rangle$:

$u=\varphi(\omega_1,\omega_2), \quad \omega_1=x_0,\ \omega_2=x_3;$

16. $\langle P_1,\ P_2,\ L_3\rangle$:

$$x_3 = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 + u,\ \omega_2 = x_0^2 - x_1^2 - x_2^2 - u^2.$$

From the invariants of the remaining two nonconjugate subalgebras it is impossible to construct the ansatzes, which reduce the eikonal equation.

Below, we present bases elements of those subalgebras and their invariants.

1. $\left\langle -X_3,\ X_4 - X_0,\ -\frac{L_3}{\lambda} - \frac{1}{2}\left(P_3 + C_3\right), 0 < \lambda < 1\right\rangle$:

   $x_0,\ (x_1^2 + x_2^2)^{1/2}$;

2. $\left\langle X_3,\ X_4 - X_0,\ \frac{L_3}{\lambda} + \frac{1}{2}\left(P_3 + C_3\right) + \frac{\alpha}{\lambda}(X_0 + X_4),\right.$

   $\alpha > 0,\ 0 < \lambda < 1\rangle$:

   $(x_1^2 + x_2^2)^{1/2},\ \alpha \arctan \frac{x_1}{x_2} - x_0.$

2.3.9 Lie Algebras of the Type $A_{3,7}^a$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,7}^a$ of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group $P(1,4)$ contains two three-dimensional nonconjugate subalgebras of the type $A_{3,7}^a$.*

Consequently, we have two ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,7}^a$.

Below, we present bases elements of those subalgebras and ansatzes corresponding with them.

1. $\langle P_1,\ P_2,\ L_3 + \lambda G, \lambda > 0 \rangle$:

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3;$$

2. $\langle P_1,\ P_2,\ L_3 + \lambda G + \alpha X_3, \alpha > 0, \lambda > 0 \rangle$:

$$\lambda x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_0^2 - x_1^2 - x_2^2 - u^2.$$

2.3.10 Lie Algebras of the Type $A_{3,8}$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,8}$ of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group $P(1,4)$ contains one three-dimensional nonconjugate subalgebra of the type $A_{3,8}$.*

Consequently, we have one ansatz, which is invariant with respect to three-dimensional nonconjugate subalgebra of the type $A_{3,8}$.

Below, we present bases elements of that subalgebra and an ansatz corresponding with it.

$\langle P_3,\ G,\ -C_3 \rangle$:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_1,\ \omega_2 = x_2.$$

2.3.11 Lie Algebras of the Type $A_{3,9}$

The results of the classification of three-dimensional nonconjugate subalgebras of the type $A_{3,9}$ of the Lie algebra of the group $P(1,4)$ can be formulated as follows:

Proposition. *The Lie algebra of the group $P(1,4)$ contains two nonconjugate subalgebras of the type $A_{3,9}$.*

Consequently, we have two ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,9}$.

Below, we present bases elements of those subalgebras and ansatzes corresponding with them.

1. $\left\langle -\frac{1}{2}\left(L_3 + \frac{1}{2}(P_3 + C_3)\right),\ \frac{1}{2}\left(L_2 + \frac{1}{2}(P_2 + C_2)\right),\ \frac{1}{2}\left(L_1 + \frac{1}{2}(P_1 + C_1)\right)\right\rangle :$

   $(x_1^2 + x_2^2 + x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = x_0;$

2. $\langle -L_3,\ -L_2,\ -L_1\rangle :$

   $u = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0,\ \omega_2 = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$

Chapter 3

Classification of symmetry reductions for the eikonal equation

The eikonal equations in the spaces of different dimensions and different types have many applications in the geometric optics, acoustics of inhomogeneous media, theories of gravity, theoretical physics, etc. The details on this theme can be found in [67–78](see also the references therein).

Those equations have also been studied by different methods. Some details can be found in [79–104] (see also the references therein).

In this chapter, we consider the eikonal equation of the form as follows:

$$\left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1,$$

where $u = u(x), \; x = (x_0, x_1, x_2, x_3) \in M(1,3)$.

In 1982, Fushchych and Shtelen [53] proved that the maximally extensive local (in sense of Lie) invariance group of this equation was a conformal group $C(1,4)$ of the $(4+1)$ -dimensional Poincaré-Minkowski space with the metric

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2, \ x_4 = u.$$

It is known that the group $C(1,4)$ contains, as a subgroup, the group $P(1,4)$.

In order to perform symmetry reduction as well as to construct classes of independent invariant solutions for this equation, we used the nonconjugate subgroups of the group $P(1,4)$.

In this chapter, we present the results of the classification of symmetry reductions of the eikonal equation for all nonconjugate subalgebras of dimensions 1, 2 and 3 of the Lie algebra of the Poincaré group $P(1,4)$.

The results are obtained using structural properties of low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ [44] as well as the results of the classification of ansatzes for the eikonal equation (see Chapter 2). Some classes of the invariant solutions for the equation under consideration are also presented.

3.1 Classification of symmetry reductions using one-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$

In this section, we present the results of the classification of symmetry reductions of the eikonal equation for all nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of dimension 1.

In Chapter 2, we have presented 19 ansatzes, which are invariant with respect to the one-dimensional nonconjugate subalgebras of the type A_1.

By now, we performed the symmetry reduction of the eikonal equation to differential equations with a fewer number of independent variables using those ansatzes. Some classes of invariant solutions are constructed.

Below, we present the results obtained.

Reduction to PDEs

1. $\langle G \rangle$:

   Ansatz

   $(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2, \omega_3),$

   $\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3.$

   Reduced equation

   $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 - 1 = 0.$

   Solution of the reduced equation

   $\varphi(\omega_1, \omega_2, \omega_3) = -(1 - c_2^2 - c_3^2)^{1/2}\omega_1 + c_2\omega_2 + c_3\omega_3 + c_1.$

   Solution of the eikonal equation

   $(x_0^2 - u^2)^{1/2} = -(1 - c_2^2 - c_3^2)^{1/2}x_1 + c_2x_2 + c_3x_3 + c_1.$

   In what follows,

   $$\varphi_i = \frac{\partial \varphi}{\partial \omega_i}, \ i = 1, 2, 3.$$

2. $\langle G + \alpha X_1, \ \alpha > 0 \rangle$:

   Ansatz

   $x_1 - \alpha \ln(x_0 + u) = \varphi(\omega_1, \omega_2, \omega_3),$

   $\omega_1 = x_2, \quad \omega_2 = x_3, \quad \omega_3 = (x_0^2 - u^2)^{1/2}.$

   Reduced equation

   $\omega_3 \left(\omega_3(\varphi_1^2 + \varphi_2^2 - \varphi_3^2 + 1) - 2\alpha\varphi_3\right) = 0.$

Solutions of the reduced equation

$$\omega_3 = 0, \quad \varphi(\omega_1, \omega_2, \omega_3) = c_1\omega_1 + c_2\omega_2 +$$
$$+\alpha \ln \left(\frac{2\alpha(\sqrt{(c_1^2 + c_2^2 + 1)\omega_3^2 + \alpha^2} + \alpha)}{\omega_3^2} \right) -$$
$$-\sqrt{(c_1^2 + c_2^2 + 1)\omega_3^2 + \alpha^2} + c_3.$$

Solutions of the eikonal equation

$$\alpha \ln \left(\frac{2\alpha \left(\sqrt{(c_1^2 + c_2^2 + 1)(x_0^2 - u^2) + \alpha^2} + \alpha \right)}{x_0 - u} \right) -$$
$$-\sqrt{(c_1^2 + c_2^2 + 1)(x_0^2 - u^2) + \alpha^2} - x_1 + c_1x_2 + c_2x_3 + c_3,$$
$$u = \pm x_0.$$

3. $\langle L_3 + \lambda G, \ \lambda > 0 \rangle$:

Ansatz

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2, \omega_3),$$
$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$
$$\omega_3 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2}.$$

Reduced equation

$$(\lambda^2\varphi\varphi_3^2 + \varphi\omega_2^2(\varphi_1^2 + \varphi_2^2 - 1) + 2\omega_2^2\varphi_3)\varphi = 0.$$

Solution of the reduced equation

$$\varphi = 0.$$

Solution of the eikonal equation

$$x_0^2 - u^2 = 0.$$

4. $\left\langle L_3 + \frac{1}{2}(P_3 + C_3) \right\rangle$:

Ansatz

$u^2 + x_3^2 = \varphi(\omega_1, \omega_2, \omega_3)$,

$\omega_1 = x_0, \quad \omega_2 = x_1^2 + x_2^2, \quad \omega_3 = \arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u}$.

Reduced equation

$\varphi \left(4\varphi\omega_2^2\varphi_2^2 - \varphi\omega_2\varphi_1^2 + 4\varphi^2\omega_2 + (\varphi + \omega_2)\varphi_3^2\right) = 0.$

Solution of the reduced equation

$\varphi = 0.$

Solution of the eikonal equation

$u^2 + x_3^2 = 0.$

5. $\left\langle L_3 + \frac{\lambda}{2}(P_3 + C_3),\ 0 < \lambda < 1 \right\rangle$:

Ansatz

$(u^2 + x_3^2)^{1/2} = \varphi(\omega_1, \omega_2, \omega_3)$,

$\omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}$,

$\omega_3 = \lambda \arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u}$.

Reduced equation

$\varphi^4(\varphi^2\omega_2^2(\varphi_1^2 - \varphi_2^2 - 1) - (\omega_2^2 + \lambda^2\varphi^2)\varphi_3^2) = 0.$

Solution of the reduced equation

$\varphi = 0.$

Solution of the eikonal equation

$u^2 + x_3^2 = 0.$

6. $\langle L_3 + \lambda G + \alpha X_3,\ \alpha > 0, \lambda > 0\rangle$:

Ansatz

$(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2, \omega_3)$,

$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \alpha \ln(x_0 + u) - \lambda x_3$,

$\omega_3 = x_3 + \alpha \arctan \dfrac{x_1}{x_2}$.

Reduced equation

$\varphi\left(\varphi\omega_1^2((\lambda\varphi_2 - \varphi_3)^2 + \varphi_1^2 - 1) + \alpha^2\varphi\varphi_3^2 + 2\alpha\omega_1^2\varphi_2\right) =$

$= 0$.

Solution of the reduced equation

$\varphi = 0$.

Solutions of the eikonal equation

$u = \pm x_0$.

7. $\left\langle L_3 + \dfrac{\lambda}{2}(P_3 + C_3) + \alpha(X_0 + X_4),\ \alpha > 0, 0 < \lambda < 1\right\rangle$:

Ansatz

$u^2 + x_3^2 = \varphi(\omega_1, \omega_2, \omega_3)$,

$\omega_1 = x_1^2 + x_2^2, \quad \omega_2 = x_0 - \alpha \arctan \dfrac{x_1}{x_2}$,

$\omega_3 = \lambda x_0 - \alpha \arctan \dfrac{x_3}{u}$.

Reduced equation

$4\varphi\omega_1^2\varphi_1^2 + (\alpha^2 - \omega_1)\varphi\varphi_2^2 - (\lambda^2\varphi - \alpha^2)\omega_1\varphi_3^2 - 2\lambda\omega_1\varphi\varphi_3\varphi_2 +$

$+ 4\omega_1\varphi^2 = 0$.

Solution of the reduced equation

$\varphi = 0.$

Solution of the eikonal equation

$u^2 + x_3^2 = 0.$

8. $\left\langle L_3 + \dfrac{1}{2}\,(P_3 + C_3) + \alpha(X_0 + X_4),\ \alpha > 0 \right\rangle :$

Ansatz

$u^2 + x_3^2 = \varphi(\omega_1, \omega_2, \omega_3),$

$\omega_1 = x_1^2 + x_2^2, \quad \omega_2 = x_0 - \alpha \arctan \dfrac{x_1}{x_2},$

$\omega_3 = x_0 - \alpha \arctan \dfrac{x_3}{u}.$

Reduced equation

$\big(4\varphi\omega_1^2\varphi_1^2 + (\alpha^2 - \omega_1)\varphi\varphi_2^2 + \omega_1(\alpha^2 - \varphi)\varphi_3^2 -$

$-2\omega_1\varphi\varphi_2\varphi_3 + 4\omega_1\varphi^2\big)\,\varphi = 0.$

Solution of the reduced equation

$\varphi = 0.$

Solution of the eikonal equation

$u^2 + x_3^2 = 0.$

It should be noted that the next results are obtained with the help of nonconjugate subalgebras of the type A_1 of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

9. $\langle L_3 \rangle$:

Ansatz

$u = \varphi(\omega_1, \omega_2, \omega_3),$

$\omega_1 = x_0, \quad \omega_2 = x_3, \quad \omega_3 = (x_1^2 + x_2^2)^{1/2}.$

Reduced equation

$\varphi_1^2 - \varphi_2^2 - \varphi_3^2 - 1 = 0.$

Solution of the reduced equation

$\varphi(\omega_1, \omega_2, \omega_3) = (c_2^2 + c_3^2 + 1)^{1/2}\omega_1 + c_2\omega_2 + c_3\omega_3 + c_1.$

Solution of the eikonal equation

$u = (c_2^2 + c_3^2 + 1)^{1/2}x_0 + c_2x_3 + c_3(x_1^2 + x_2^2)^{1/2} + c_1.$

10. $\langle L_3 - P_3 \rangle$:

Ansatz

$\arctan\frac{x_1}{x_2} + \frac{x_3}{x_0 + u} = \varphi(\omega_1, \omega_2, \omega_3),$

$\omega_1 = x_0 + u,\ \omega_2 = (x_1^2 + x_2^2)^{1/2},\ \omega_3 = (x_0^2 - x_3^2 - u^2)^{1/2}.$

Reduced equation

$\omega_3\omega_1^2\left(2\omega_1^3\omega_2^2\varphi_1\varphi_3 - \omega_1^2\omega_2^2\omega_3(\varphi_2^2 - \varphi_3^2) - \right.$

$\left. -(\omega_1^2 + \omega_2^2)\omega_3\right) = 0.$

Solutions of the reduced equation

$\omega_1 = 0, \quad \omega_3 = 0.$

Solutions of the eikonal equation

$u = -x_0, \quad x_0^2 - x_3^2 - u^2 = 0.$

11. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0\rangle$:

Ansatz

$u = \varphi(\omega_1, \omega_2, \omega_3),$

$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = x_0 - \alpha \arctan \dfrac{x_1}{x_2}.$

Reduced equation

$\omega_2^2(\varphi_1^2 + \varphi_2^2) + (\alpha^2 - \omega_2^2)\varphi_3^2 + \omega_2^2 = 0.$

Solution of the reduced equation

$$\varphi(\omega_1, \omega_2, \omega_3) = -i((c_1^2 - c_3^2 + 1)\omega_2^2 + c_3^2\alpha^2)^{1/2} + c_3\omega_3 +$$
$$+ i\alpha c_3 \operatorname{arctanh}\left(\frac{c_3\alpha}{((c_1^2 - c_3^2 + 1)\omega_2^2 + c_3^2\alpha^2)^{1/2}}\right) + c_1\omega_1 +$$
$$+ c_4.$$

Solution of the eikonal equation

$$u = i\alpha c_3 \operatorname{arctanh} \frac{c_3\alpha}{((c_1^2 - c_3^2 + 1)(x_1^2 + x_2^2) + c_3^2\alpha^2)^{1/2}} -$$
$$-i\left((c_1^2 - c_3^2 + 1)(x_1^2 + x_2^2) + c_3^2\alpha^2\right)^{1/2} +$$
$$+ c_3\left(x_0 - \alpha \arctan \frac{x_1}{x_2}\right) + c_1x_3 + c_4.$$

12. $\langle L_3 + \alpha X_3,\ \alpha > 0\rangle$:

Ansatz

$u = \varphi(\omega_1, \omega_2, \omega_3),$

$\omega_1 = x_0, \quad \omega_2 = x_1^2 + x_2^2, \quad \omega_3 = x_3 + \alpha \arctan \dfrac{x_1}{x_2}.$

Reduced equation

$\omega_2\varphi_1^2 - 4\omega_2^2\varphi_2^2 - (\alpha^2 + \omega_2)\varphi_3^2 - \omega_2 = 0.$

Solution of the reduced equation

$$\varphi(\omega_1,\omega_2,\omega_3) = ((c_1^2 - c_3^2 - 1)\omega_2 - c_3^2\alpha^2)^{1/2} -$$

$$-\alpha c_3 \arctan \frac{((c_1^2 - c_3^2 - 1)\omega_2 - c_3^2\alpha^2)^{1/2}}{c_3\alpha} +$$

$$+c_1\omega_1 + c_3\omega_3 + c_4.$$

Solution of the eikonal equation

$$u = \sqrt{(c_1^2 - c_3^2 - 1)(x_1^2 + x_2^2) - \alpha^2 c_3^2} + c_3\alpha \arctan \frac{x_1}{x_2} -$$

$$-c_3\alpha \arctan \left(\frac{\sqrt{(c_1^2 - c_3^2 - 1)(x_1^2 + x_2^2) - \alpha^2 c_3^2}}{c_3\alpha} \right) +$$

$$+c_3 x_3 + c_1 x_0 + c_4.$$

13. $\langle L_3 + 2X_4 \rangle$:

Ansatz

$$x_0 - u + 2\arctan \frac{x_2}{x_1} = \varphi(\omega_1,\omega_2,\omega_3),$$

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_3 = x_3.$$

Reduced equation

$$\omega_2^2(4\varphi_1 + \varphi_2^2 + \varphi_3^2) + 4 = 0.$$

Solution of the reduced equation

$$\varphi(\omega_1,\omega_2,\omega_3) = 2i\sqrt{(c_1 + c_3^2)\omega_2^2 + 1} -$$

$$-2i \operatorname{arctanh} \left(\sqrt{(c_1 + c_3^2)\omega_2^2 + 1} \right) + c_1\omega_1 + 2c_3\omega_3 + c_4.$$

Solution of the eikonal equation

$$x_0 - u + 2\arctan \frac{x_2}{x_1} = 2i\sqrt{(c_3^2 + c_1)(x_1^2 + x_2^2) + 1} -$$

$$-2i \operatorname{arctanh} \left(\sqrt{(c_3^2 + c_1)(x_1^2 + x_2^2) + 1} \right) + c_1(x_0 + u) +$$

$+c_3x_3+c_4.$

14. $\langle L_3-P_3+2\alpha X_0,\ \alpha>0\rangle:$

Ansatz

$(x_0+u)^2+4x_3\alpha=\varphi(\omega_1,\omega_2,\omega_3),$

$\omega_1=(x_1^2+x_2^2)^{1/2},\quad \omega_2=2\alpha\arctan\dfrac{x_1}{x_2}-x_0-u,$

$\omega_3=2(x_0+u)^3+12\alpha^2(x_0-u)+12\alpha x_3(x_0+u).$

Reduced equation

$\omega_1^2\varphi_1^2+4\alpha^2\varphi_2^2-144\alpha^2\varphi\omega_1^2\varphi_3^2+48\alpha^2\omega_1^2\varphi_2\varphi_3+16\alpha^2\omega_1^2=0.$

15. $\langle P_3\rangle:$

Ansatz

$(x_0^2-x_3^2-u^2)^{1/2}=\varphi(\omega_1,\omega_2,\omega_3),$

$\omega_1=x_1,\quad \omega_2=x_2,\quad \omega_3=x_0+u.$

Reduced equation

$\varphi(\varphi\varphi_1^2+\varphi\varphi_2^2+2\omega_3\varphi_3-\varphi)=0.$

Solution of the reduced equation

$\varphi=0.$

Solution of the eikonal equation

$x_0^2-x_3^2-u^2=0.$

16. $\langle P_3-2X_0\rangle:$

Ansatz

$\dfrac{1}{6}(x_0+u)^3+x_3(x_0+u)+x_0-u=\varphi(\omega_1,\omega_2,\omega_3),$

$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = (x_0 + u)^2 + 4x_3.$

Reduced equation

$\varphi_1^2 + \varphi_2^2 + 16\varphi_3^2 - \omega_3 = 0.$

Solution of the reduced equation

$\varphi(\omega_1, \omega_2, \omega_3) = c_1\omega_1 + c_2\omega_2 - \frac{1}{6}(\omega_3 - c_1^2 - c_2^2)^{3/2} + c_3.$

Solution of the eikonal equation

$\frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u =$

$= -\frac{1}{6}\left((x_0 + u)^2 + 4x_3 - c_1^2 - c_2^2\right)^{3/2} + c_1x_1 + c_2x_2 + c_3.$

17. $\langle P_3 - X_1 \rangle :$

Ansatz

$x_1(x_0 + u) - x_3 = \varphi(\omega_1, \omega_2, \omega_3),$

$\omega_1 = x_2, \quad \omega_2 = x_0 + u, \quad \omega_3 = x_3^2 + 2u(x_0 + u).$

Reduced equation

$\varphi_1^2 + 4\omega_2\varphi_2\varphi_3 + 4(\omega_2^2 + \omega_3)\varphi_3^2 - 4\varphi\varphi_3 + \omega_2^2 + 1 = 0.$

18. $\langle X_0 + X_4 \rangle :$

Ansatz

$u = \varphi(\omega_1, \omega_2, \omega_3),$

$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3.$

Reduced equation

$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + 1 = 0.$

Solution of the reduced equation

$$\varphi(\omega_1, \omega_2, \omega_3) = i(c_2^2 + c_3^2 + 1)^{1/2}\omega_1 + c_2\omega_2 + c_3\omega_3 + c_1.$$

Solution of the eikonal equation

$$u = i(c_2^2 + c_3^2 + 1)^{1/2}x_1 + c_2x_2 + c_3x_3 + c_1.$$

In one case, the reduced equation is a two-dimensional PDE.

19. $\langle X_4 \rangle$:

Ansatz

$$x_3 = \varphi(\omega_1, \omega_2, \omega_3),$$

$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_0 + u.$$

Reduced equation

$$\varphi_1^2 + \varphi_2^2 + 1 = 0.$$

Solution of the reduced equation

$$\varphi(\omega_1, \omega_2, \omega_3) = -i(c_2^2 + 1)^{1/2}\omega_1 + c_2\omega_2 + c_1 + f(\omega_3).$$

Solution of the eikonal equation

$$x_3 = -i(c_2^2 + 1)^{1/2}x_1 + c_2x_2 + c_1 + f(x_0 + u),$$

where f is an arbitrary smooth function.

There are no reductions

From the invariants of the remaining one nonconjugate subalgebra it is impossible to construct an ansatz, which reduces the eikonal equation. More details on this theme can be found in Chapter 2.

3.2 Classification of symmetry reductions using two-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$

In this section, we present the results of the classification of symmetry reductions of the eikonal equation for all nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of dimension 2.

3.2.1 Lie Algebras of the Type $2A_1$

In Chapter 2, we have presented 37 ansatzes, which are invariant with respect to the two-dimensional nonconjugate subalgebras of the type $2A_1$.

By now, we performed the symmetry reduction of the eikonal equation to differential equations with a fewer number of independent variables using those ansatzes. Some classes of invariant solutions are constructed.

Below, we present the results obtained.

Reduction to PDEs

1. $\langle G \rangle \oplus \langle L_3 \rangle$:

   Ansatz

   $(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$

   $\omega_1 = x_3,\ \omega_2 = (x_1^2 + x_2^2)^{1/2}.$

   Reduced equation

   $\varphi_1^2 + \varphi_2^2 - 1 = 0.$

   Solution of the reduced equation

   $\varphi(\omega_1, \omega_2) = (1 - c_2^2)^{1/2}\omega_1 + c_2\omega_2 + c_1.$

Solution of the eikonal equation

$(x_0^2 - u^2)^{1/2} = (1 - c_2^2)^{1/2} x_3 + c_2 (x_1^2 + x_2^2)^{1/2} + c_1.$

In what follows,

$$\varphi_i = \frac{\partial \varphi}{\partial \omega_i}, \; i = 1, 2.$$

2. $\langle G + \alpha X_3, \; \alpha > 0 \rangle \oplus \langle L_3 \rangle :$

Ansatz

$x_3 - \alpha \ln(x_0 + u) = \varphi(\omega_1, \omega_2),$

$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0^2 - u^2)^{1/2}.$

Reduced equation

$\omega_2 \left(\omega_2 (\varphi_1^2 - \varphi_2^2 + 1) - 2\alpha \varphi_2 \right) = 0.$

Solutions of the reduced equation

$$\varphi(\omega_1, \omega_2) = \alpha \ln \left(\frac{2\alpha \left(\sqrt{(c_1^2 + 1)\omega_2^2 + \alpha^2} + \alpha \right)}{\omega_2^2} \right) -$$

$$-\sqrt{(c_1^2 + 1)\omega_2^2 + \alpha^2} + c_1 \omega_1 + c_2, \quad \omega_2 = 0.$$

Solutions of the eikonal equation

$$x_3 - \alpha \ln(x_0 + u) = -\sqrt{(c_1^2 + 1)(x_0^2 - u^2) + \alpha^2} +$$

$$+\alpha \ln \left(\frac{2\alpha \left(\sqrt{(c_1^2 + 1)(x_0^2 - u^2) + \alpha^2} + \alpha \right)}{x_0^2 - u^2} \right) +$$

$$+c_1 \sqrt{x_1^2 + x_2^2} + c_2, \quad x_0^2 - u^2 = 0.$$

3. $\langle G\rangle \oplus \langle L_3 + \alpha X_3,\ \alpha > 0\rangle$:

Ansatz

$$x_3 + \alpha \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0^2 - u^2)^{1/2}.$$

Reduced equation

$$(\varphi_1^2 - \varphi_2^2 + 1)\omega_1^2 + \alpha^2 = 0.$$

Solution of the reduced equation

$$\varphi(\omega_1, \omega_2) = \alpha \arctan \frac{\alpha}{\sqrt{(c_2^2 - 1)\omega_1^2 - \alpha^2}} +$$

$$+ \sqrt{(c_2^2 - 1)\omega_1^2 - \alpha^2} + c_2\omega_2 + c_1.$$

Solution of the eikonal equation

$$x_3 + \alpha \arctan \frac{x_1}{x_2} = \alpha \arctan \frac{\alpha}{\sqrt{(c_2^2 - 1)(x_1^2 + x_2^2) - \alpha^2}} +$$

$$+ c_2(x_0^2 - u^2)^{1/2} + \sqrt{(c_2^2 - 1)(x_1^2 + x_2^2) - \alpha^2} + c_1.$$

4. $\langle G + \alpha X_3,\ \alpha > 0\rangle \oplus \langle L_3 + \beta X_3,\ \beta > 0\rangle$:

Ansatz

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_3 - \alpha \ln(x_0 + u) - \beta \arctan \frac{x_2}{x_1}, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$\varphi \left(\varphi\omega_2^2(\varphi_1^2 + \varphi_2^2 - 1) - 2\alpha\omega_2^2\varphi_1 + \beta^2\varphi\varphi_1^2\right) = 0.$$

Solution of the reduced equation

$$\varphi = 0.$$

Solution of the eikonal equation

$x_0^2 - u^2 = 0.$

5. $\langle L_3 + \lambda G,\ \lambda > 0\rangle \oplus \langle X_3\rangle :$

Ansatz

$(x_1^2 + x_2^2)^{1/2} = \varphi(\omega_1, \omega_2),$

$\omega_1 = (x_0^2 - u^2)^{1/2}, \quad \omega_2 = \ln(x_0 + u) + \lambda \arctan \dfrac{x_1}{x_2}.$

Reduced equation

$\omega_1(\lambda^2\omega_1\varphi_2^2 - \varphi^2\omega_1\varphi_1^2 - 2\varphi^2\varphi_1\varphi_2 + \varphi^2\omega_1) = 0.$

Solution of the reduced equation

$\omega_1 = 0.$

Solution of the eikonal equation

$x_0^2 - u^2 = 0.$

6. $\langle G\rangle \oplus \langle X_1\rangle :$

Ansatz

$(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$

$\omega_1 = x_2, \quad \omega_2 = x_3.$

Reduced equation

$\varphi_1^2 + \varphi_2^2 - 1 = 0.$

Solution of the reduced equation

$\varphi(\omega_1, \omega_2) = (1 - c_2^2)^{1/2}\omega_1 + c_2\omega_2 + c_1.$

Solution of the eikonal equation

$(x_0^2 - u^2)^{1/2} = (1 - c_2^2)^{1/2}x_2 + c_2x_3 + c_1.$

7. $\langle G+\alpha X_2,\ \alpha>0\rangle \oplus \langle X_1\rangle$:

Ansatz

$x_3=\varphi(\omega_1,\omega_2),$

$\omega_1=(x_0^2-u^2)^{1/2}, \quad \omega_2=x_2-\alpha\ln(x_0+u).$

Reduced equation

$\omega_1\left((\varphi_1^2-\varphi_2^2-1)\omega_1-2\alpha\varphi_1\varphi_2\right)=0.$

Solutions of the reduced equation

$$\omega_1=0,\ \varphi(\omega_1,\omega_2)=\alpha c_2\,\mathrm{arctanh}\,\frac{\alpha c_2}{\sqrt{(c_2^2+1)\omega_1^2+c_2^2\alpha^2}}+$$
$$+\alpha c_2\ln(\omega_1)+c_2\omega_2-\sqrt{(c_2^2+1)\omega_1^2+c_2^2\alpha^2}+c_1.$$

Solutions of the eikonal equation

$$x_0^2-u^2=0, \quad x_3+\sqrt{(c_2^2+1)(x_0^2-u^2)+\alpha^2c_2^2}=$$
$$=\alpha c_2\,\mathrm{arctanh}\,\frac{\alpha c_2}{\sqrt{(c_2^2+1)(x_0^2-u^2)+\alpha^2c_2^2}}+$$
$$+\frac{\alpha c_2}{2}\ln\frac{x_0-u}{x_0+u}+c_2x_2+c_1.$$

8. $\langle L_3\rangle \oplus \langle P_3+C_3\rangle$:

Ansatz

$(u^2+x_3^2)^{1/2}=\varphi(\omega_1,\omega_2),$

$\omega_1=x_0, \quad \omega_2=(x_1^2+x_2^2)^{1/2}.$

Reduced equation

$(\varphi_1^2-\varphi_2^2-1)\varphi^2=0.$

Solutions of the reduced equation

$\varphi(\omega_1,\omega_2)=(c_2^2+1)^{1/2}\omega_1+c_2\omega_2+c_1, \quad \varphi=0.$

Solutions of the eikonal equation

$(u^2 + x_3^2)^{1/2} = (c_2^2 + 1)^{1/2} x_0 + c_2 (x_1^2 + x_2^2)^{1/2} + c_1,$

$u^2 + x_3^2 = 0.$

9. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0\rangle \oplus \langle P_3 + C_3\rangle :$

Ansatz

$x_0 - \alpha \arctan \dfrac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$

$\omega_1 = (x_1^2 + x_2^2)^{1/2},\ \omega_2 = (u^2 + x_3^2)^{1/2}.$

Reduced equation

$((\varphi_1^2 + \varphi_2^2 - 1)\omega_1^2 + \alpha^2)\omega_2^2 = 0.$

Solutions of the reduced equation

$\varphi(\omega_1, \omega_2) = \sqrt{(1 - c_2^2)\omega_1^2 - \alpha^2} +$

$+\alpha \arctan \dfrac{\alpha}{\sqrt{(1 - c_2^2)\omega_1^2 - \alpha^2}} + c_2\omega_2 + c_1,\ \omega_2 = 0.$

Solutions of the eikonal equation

$x_0 - \alpha \arctan \dfrac{x_1}{x_2} = \alpha \arctan \dfrac{\alpha}{\sqrt{(1 - c_2^2)(x_1^2 + x_2^2) - \alpha^2}} +$

$+c_2\sqrt{u^2 + x_3^2} + \sqrt{(1 - c_2^2)(x_1^2 + x_2^2) - \alpha^2} + c_1,$

$u^2 + x_3^2 = 0.$

10. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0\rangle \oplus \langle P_3 + C_3 + 2\beta(X_0 + X_4),$
$\beta > 0\rangle :$

Ansatz

$\alpha \arctan \dfrac{x_1}{x_2} + \beta \arctan \dfrac{x_3}{u} - x_0 = \varphi(\omega_1, \omega_2),$

$\omega_1 = (x_1^2 + x_2^2)^{1/2},\ \omega_2 = (u^2 + x_3^2)^{1/2}.$

Reduced equation

$\omega_2^4\left(\omega_1^2\omega_2^2(\varphi_1^2+\varphi_2^2-1)+\beta^2\omega_1^2+\alpha^2\omega_2^2\right)=0.$

Solution of the reduced equation

$\omega_2=0.$

Solution of the eikonal equation

$u^2+x_3^2=0.$

11. $\left\langle L_3+\frac{1}{2}(P_3+C_3)\right\rangle\oplus\langle X_0+X_4\rangle:$

Ansatz

$\arctan\frac{x_1}{x_2}-\arctan\frac{x_3}{u}=\varphi(\omega_1,\omega_2),$

$\omega_1=(x_1^2+x_2^2)^{1/2},\quad \omega_2=(u^2+x_3^2)^{1/2}.$

Reduced equation

$\omega_2^4\left(\omega_1^2\omega_2^2(\varphi_1^2+\varphi_2^2)+\omega_1^2+\omega_2^2\right)=0.$

Solution of the reduced equation

$\omega_2=0.$

Solution of the eikonal equation

$u^2+x_3^2=0.$

12. $\left\langle L_3+\frac{\lambda}{2}(P_3+C_3),\ 0<\lambda<1\right\rangle\oplus\langle X_0+X_4\rangle:$

Ansatz

$\lambda\arctan\frac{x_1}{x_2}-\arctan\frac{x_3}{u}=\varphi(\omega_1,\omega_2),$

$\omega_1=(x_1^2+x_2^2)^{1/2},\quad \omega_2=(u^2+x_3^2)^{1/2}.$

Reduced equation

$\omega_2^4\left(\omega_1^2\omega_2^2(\varphi_1^2+\varphi_2^2)+\omega_1^2+\lambda^2\omega_2^2\right)=0.$

Solution of the reduced equation

$\omega_2=0.$

Solution of the eikonal equation

$u^2+x_3^2=0.$

It should be noted that the next results are obtained with the help of nonconjugate subalgebras of the type $2A_1$ of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)\subset P(1,4)$.

13. $\langle L_3\rangle\oplus\langle P_3\rangle$:

Ansatz

$(x_0^2-x_3^2-u^2)^{1/2}=\varphi(\omega_1,\omega_2),$

$\omega_1=x_0+u,\quad \omega_2=(x_1^2+x_2^2)^{1/2}.$

Reduced equation

$\varphi(\varphi\varphi_2^2+2\omega_1\varphi_1-\varphi)=0.$

Solution of the reduced equation

$\varphi=0.$

Solution of the eikonal equation

$x_0^2-x_3^2-u^2=0.$

14. $\langle L_3-P_3\rangle\oplus\langle X_4\rangle$:

Ansatz

$x_0+u=\varphi(\omega_1,\omega_2),$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \arctan\frac{x_1}{x_2} + \frac{x_3}{x_0 + u}.$$

Reduced equation

$$\varphi^2(\varphi^2\omega_1^2\varphi_1^2 + \varphi^2\varphi_2^2 + \omega_1^2\varphi_2^2) = 0.$$

Solution of the reduced equation

$$\varphi = 0.$$

Solution of the eikonal equation

$$u = -x_0.$$

15. $\langle L_3 \rangle \oplus \langle X_0 + X_4 \rangle$:

Ansatz

$$u = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_3.$$

Reduced equation

$$\varphi_1^2 + \varphi_2^2 + 1 = 0.$$

Solution of the reduced equation

$$\varphi(\omega_1, \omega_2) = i(c_2^2 + 1)^{1/2}\omega_1 + c_2\omega_2 + c_1.$$

Solution of the eikonal equation

$$u = i(c_2^2 + 1)^{1/2}(x_1^2 + x_2^2)^{1/2} + c_2x_3 + c_1.$$

16. $\langle L_3 + \alpha(X_0 + X_4),\ \alpha > 0 \rangle \oplus \langle X_4 \rangle$:

Ansatz

$$x_0 + u - \alpha\arctan\frac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$\omega_2^2(\varphi_1^2+\varphi_2^2)+\alpha^2=0.$

Solution of the reduced equation

$$\varphi(\omega_1,\omega_2)=c_1\omega_1+i\sqrt{c_1^2\omega_2^2+\alpha^2}-$$
$$-i\alpha\,\mathrm{arctanh}\,\frac{\alpha}{\sqrt{c_1^2\omega_2^2+\alpha^2}}+c_2.$$

Solution of the eikonal equation

$$u=\alpha\arctan\frac{x_1}{x_2}+i\sqrt{c_1^2(x_1^2+x_2^2)+\alpha^2}-$$
$$-i\alpha\,\mathrm{arctanh}\,\frac{\alpha}{\sqrt{c_1^2(x_1^2+x_2^2)+\alpha^2}}-x_0+c_1x_3+c_2.$$

17. $\langle L_3+\alpha X_3,\ \alpha>0\rangle\oplus\langle X_0+X_4\rangle:$

Ansatz

$$x_3+\alpha\arctan\frac{x_1}{x_2}=\varphi(\omega_1,\omega_2),$$
$$\omega_1=u,\quad \omega_2=(x_1^2+x_2^2)^{1/2}.$$

Reduced equation

$(\varphi_1^2+\varphi_2^2+1)\omega_2^2+\alpha^2=0.$

Solution of the reduced equation

$$\varphi(\omega_1,\omega_2)=i\alpha\,\mathrm{arctanh}\,\frac{\alpha}{\sqrt{(c_1^2+1)\omega_2^2+\alpha^2}}-$$
$$-i\sqrt{(c_1^2+1)\omega_2^2+\alpha^2}+c_1\omega_1+c_2.$$

Solution of the eikonal equation

$$u=\frac{\alpha}{c_1}\arctan\left(\frac{x_1\sqrt{(c_1^2+1)(x_1^2+x_2^2)+\alpha^2}-i\alpha x_2}{x_2\sqrt{(c_1^2+1)(x_1^2+x_2^2)+\alpha^2}+i\alpha x_1}\right)+$$

$$+\frac{i}{c_1}\sqrt{(c_1^2+1)(x_1^2+x_2^2)+\alpha^2}+\frac{x_3}{c_1}+c_2.$$

18. $\langle L_3+2X_4\rangle \oplus \langle X_3\rangle$:

Ansatz

$$x_0-u+2\arctan\frac{x_2}{x_1}=\varphi(\omega_1,\omega_2),$$

$$\omega_1=x_0+u, \quad \omega_2=(x_1^2+x_2^2)^{1/2}.$$

Reduced equation

$$(\varphi_2^2+4\varphi_1)\omega_2^2+4=0.$$

Solution of the reduced equation

$$\varphi(\omega_1,\omega_2)=c_1\omega_1-2i\sqrt{c_1\omega_2^2+1}+$$

$$+2i\,\mathrm{arctanh}\,\frac{1}{\sqrt{c_1\omega_2^2+1}}+c_2.$$

Solution of the eikonal equation

$$x_0-u+2\arctan\frac{x_2}{x_1}=2i\,\mathrm{arctanh}\,\frac{1}{\sqrt{c_1(x_1^2+x_2^2)+1}}-$$

$$-2i\sqrt{c_1(x_1^2+x_2^2)+1}+c_1(x_0+u)+c_2.$$

19. $\langle L_3-P_3+2\alpha X_0,\ \alpha\neq 0\rangle \oplus \langle X_4\rangle$:

Ansatz

$$x_0+u-2\alpha\arctan\frac{x_1}{x_2}=\varphi(\omega_1,\omega_2),$$

$$\omega_1=(x_1^2+x_2^2)^{1/2}, \quad \omega_2=(x_0+u)^2+4\alpha x_3.$$

Reduced equation

$$\omega_1^2(\varphi_1^2+16\alpha^2\varphi_2^2)+4\alpha^2=0.$$

Solution of the reduced equation

$$\varphi(\omega_1,\omega_2) = 2i\alpha\sqrt{4c_2^2\omega_1^2+1} -$$
$$-2i\alpha \operatorname{arctanh}\left(\frac{1}{\sqrt{4c_2^2\omega_1^2+1}}\right) + c_2\omega_2 + c_1.$$

Solution of the eikonal equation

$$x_0 + u - 2\alpha\arctan\frac{x_1}{x_2} = 2i\alpha\sqrt{4c_2^2(x_1^2+x_2^2)+1} -$$
$$-2i\alpha \operatorname{arctanh}\frac{1}{\sqrt{4c_2^2(x_1^2+x_2^2)+1}} +$$
$$+c_2\left((x_0+u)^2 + 4\alpha x_3\right) + c_1.$$

20. $\langle L_3 + 2X_4\rangle \oplus \langle P_3 - 2\beta X_0,\ \beta > 0\rangle$:

Ansatz

$$(x_0+u)^2 + 4\beta x_3 = \varphi(\omega_1,\omega_2),$$
$$\omega_1 = (x_1^2+x_2^2)^{1/2},$$
$$\omega_2 = 4\beta\arctan\frac{x_1}{x_2} - 2\beta(x_0-u) - \frac{1}{3\beta}(x_0+u)^3 -$$
$$-2x_3(x_0+u).$$

Reduced equation

$$\omega_1^2\varphi_1^2 + 4(4\beta^2 - \varphi\omega_1^2)\varphi_2^2 + 16\beta^2\omega_1^2 = 0.$$

21. $\langle L_3 + 2X_4\rangle \oplus \langle P_3\rangle$:

Ansatz

$$\frac{x_3^2}{x_0+u} + 2\arctan\frac{x_1}{x_2} + 2u = \varphi(\omega_1,\omega_2),$$
$$\omega_1 = (x_1^2+x_2^2)^{1/2},\ \omega_2 = x_0 + u.$$

Reduced equation

$$\omega_2^4\left((\varphi_1^2 - 4\varphi_2 + 4)\omega_1^2 + 4\right) = 0.$$

Solutions of the reduced equation

$$\omega_2 = 0, \quad \varphi(\omega_1, \omega_2) = 2\sqrt{(c_2 - 1)\omega_1^2 - 1} +$$
$$+2\arctan\frac{1}{\sqrt{(c_2 - 1)\omega_1^2 - 1}} + c_2\omega_2 + c_1.$$

Solutions of the eikonal equation

$$u = -x_0, \quad \frac{x_3^2}{x_0 + u} + 2\arctan\frac{x_1}{x_2} + 2u =$$
$$= 2\sqrt{(c_2 - 1)(x_1^2 + x_2^2) - 1} +$$
$$+2\arctan\left(\frac{1}{\sqrt{(c_2 - 1)(x_1^2 + x_2^2) - 1}}\right) + c_2(x_0 + u) + c_1.$$

22. $\langle L_3\rangle \oplus \langle P_3 - 2X_0\rangle$:

Ansatz

$$\frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \ \omega_2 = (x_0 + u)^2 + 4x_3.$$

Reduced equation

$$\varphi_1^2 + 16\varphi_2^2 - \omega_2 = 0.$$

Solution of the reduced equation

$$\varphi(\omega_1, \omega_2) = c_1\omega_1 - \frac{1}{6}(\omega_2 - c_1^2)^{3/2} + c_2.$$

Solution of the eikonal equation

$$\frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u =$$

$$= c_1\sqrt{x_1^2 + x_2^2} - \frac{1}{6}\left((x_0+u)^2 + 4x_3 - c_1^2\right)^{3/2} + c_2.$$

23. $\langle P_1\rangle \oplus \langle P_2\rangle$:

Ansatz

$x_0 + u = \varphi(\omega_1, \omega_2)$,

$\omega_1 = x_3, \quad \omega_2 = (x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2}$.

Reduced equation

$\omega_2(\omega_2\varphi_1^2 - \omega_2\varphi_2^2 + 2\varphi\varphi_2) = 0.$

Solution of the reduced equation

$\omega_2 = 0.$

Solution of the eikonal equation

$x_0^2 - x_1^2 - x_2^2 - u^2 = 0.$

24. $\langle P_1 - X_3\rangle \oplus \langle P_2\rangle$:

Ansatz

$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2)$,

$\omega_1 = x_0 + u, \quad \omega_2 = x_3 - \dfrac{x_1}{x_0 + u}$.

Reduced equation

$\omega_1^2\varphi\left((\varphi\omega_1^2 + \varphi)\varphi_2^2 + 2\omega_1^3\varphi_1 - \varphi\omega_1^2\right) = 0.$

Solutions of the reduced equation

$\omega_1 = 0, \quad \varphi = 0.$

Solutions of the eikonal equation

$u = -x_0, \quad x_0^2 - x_1^2 - x_2^2 - u^2 = 0.$

25. $\langle P_1 - X_3\rangle \oplus \langle P_2 - \gamma X_2 - \beta X_3,\ \beta > 0, \gamma > 0\rangle$:

Ansatz

$$\frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u+\gamma} + 2u = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_0 + u, \quad \omega_2 = \frac{\beta x_2}{x_0+u+\gamma} + \frac{x_1}{x_0+u} - x_3.$$

Reduced equation

$$(\omega_1+\gamma)^2\omega_1^2\left((\omega_1+\gamma)^2\left(4(1-\varphi_1)\omega_1^2 + (\omega_1^2+1)\varphi_2^2\right) + \right.$$
$$\left. +\beta^2\omega_1^2\varphi_2^2\right) = 0.$$

Solutions of the reduced equation

$$\omega_1 = 0,\ \omega_1+\gamma = 0,\ \varphi(\omega_1,\omega_2) = -\frac{c_2^2}{\omega_1}\times\frac{(\beta^2+1)\omega_1+\gamma}{\omega_1+\gamma} +$$
$$+(c_2^2+1)\omega_1 + 2c_2\omega_2 + c_1.$$

Solutions of the eikonal equation

$$u = -x_0, \quad u = -x_0-\gamma, \quad \frac{x_1(x_1-2c_2)}{x_0+u} +$$
$$+\frac{(x_2-\beta c_2)^2 + c_2^2}{x_0+u+\gamma} + \frac{\gamma c_2^2}{(x_0+u)(x_0+u+\gamma)} -$$
$$-(c_2^2+1)(x_0+u) + 2c_2x_3 + 2u + c_1 = 0.$$

26. $\langle P_1 - X_3\rangle \oplus \langle P_2 - \gamma X_2,\ \gamma > 0\rangle$:

Ansatz

$$\frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u+\gamma} + 2u = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_0 + u,\ \omega_2 = \frac{x_1}{x_0+u} - x_3.$$

Reduced equation

$$\omega_1^2(\omega_1+\gamma)^4\left((\omega_1^2+1)\varphi_2^2+4\omega_1^2(1-\varphi_1)\right)=0.$$

Solutions of the reduced equation

$$\omega_1=0,\quad \omega_1+\gamma=0,$$

$$\varphi(\omega_1,\omega_2)=(c_2^2+1)\omega_1-\frac{c_2^2}{\omega_1}+2c_2\omega_2+c_1.$$

Solutions of the eikonal equation

$$u=-x_0,\quad u=-x_0-\gamma,\quad \frac{(x_1-c_2)^2}{x_0+u}+\frac{x_2^2}{x_0+u+\gamma}+$$
$$+2u=(c_2^2+1)(x_0+u)-2c_2x_3+c_1.$$

27. $\langle P_1\rangle\oplus\langle P_2-X_2-\beta X_3,\ \beta>0\rangle:$

Ansatz

$$\frac{x_1^2}{x_0+u}+\frac{x_2^2}{x_0+u+1}+2u=\varphi(\omega_1,\omega_2),$$
$$\omega_1=x_0+u,\ \omega_2=x_3-\beta\frac{x_2}{x_0+u+1}.$$

Reduced equation

$$(\omega_1+1)^2\omega_1^4\left(((\omega_1+1)^2+\beta^2)\varphi_2^2-\right.$$
$$\left.-4(\omega_1+1)^2(\varphi_1-1)\right)=0.$$

Solutions of the reduced equation

$$\omega_1+1=0,\quad \omega_1=0,$$

$$\varphi(\omega_1,\omega_2)=\left(\frac{c_2^2}{4}+1\right)\omega_1+c_2\omega_2-\frac{\beta^2c_2^2}{4}(\omega_1+1)+c_1.$$

Solutions of the eikonal equation

$$u=-1-x_0, \quad u=-x_0, \quad \frac{x_1^2}{x_0+u}+2u=\left(\frac{c_2^2}{4}+1\right)\times$$

$$\times(x_0+u)-\frac{(\beta c_2+2x_2)^2}{4(x_0+u+1)}+c_2x_3+c_1.$$

28. $\langle P_1\rangle \oplus \langle P_2-X_2\rangle:$

Ansatz

$$\frac{x_1^2}{x_0+u}+\frac{x_2^2}{x_0+u+1}+2u=\varphi(\omega_1,\omega_2),$$

$$\omega_1=x_0+u,\ \omega_2=x_3.$$

Reduced equation

$$(\omega_1+1)^4\omega_1^4(\varphi_2^2-4\varphi_1+4)=0.$$

Solutions of the reduced equation

$$\omega_1+1=0, \quad \omega_1=0,$$

$$\varphi(\omega_1,\omega_2)=\left(\frac{c_2^2}{4}+1\right)\omega_1+c_2\omega_2+c_1.$$

Solutions of the eikonal equation

$$u=-1-x_0, \quad u=-x_0, \quad \frac{x_1^2}{x_0+u}+\frac{x_2^2}{x_0+u+1}+2u=$$

$$=\left(\frac{c_2^2}{4}+1\right)(x_0+u)+c_2x_3+c_1.$$

29. $\langle P_3\rangle \oplus \langle X_1\rangle:$

Ansatz

$$(x_0^2-x_3^2-u^2)^{1/2}=\varphi(\omega_1,\omega_2),$$

$$\omega_1=x_2,\ \omega_2=x_0+u.$$

Reduced equation

$\varphi(\varphi\varphi_1^2 + 2\omega_2\varphi_2 - \varphi) = 0.$

Solution of the reduced equation

$\varphi = 0.$

Solution of the eikonal equation

$x_0^2 - x_3^2 - u^2 = 0.$

30. $\langle P_3 - X_2 \rangle \oplus \langle X_1 \rangle$:

Ansatz

$$x_2 - \frac{x_3}{x_0 + u} = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_0 + u, \quad \omega_2 = (x_0^2 - x_3^2 - u^2)^{1/2}.$$

Reduced equation

$\omega_1^2\omega_2(2\omega_1^3\varphi_1\varphi_2 + \omega_1^2\omega_2(\varphi_2^2 - 1) - \omega_2) = 0.$

Solutions of the reduced equation

$\omega_1 = 0, \quad \omega_2 = 0.$

Solutions of the eikonal equation

$u = -x_0, \quad x_0^2 - x_3^2 - u^2 = 0.$

31. $\langle P_3 - 2X_0 \rangle \oplus \langle X_4 \rangle$:

Ansatz

$$(x_0 + u)^2 + 4x_3 = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_1, \quad \omega_2 = x_2.$$

Reduced equation

$\varphi_1^2 + \varphi_2^2 + 16 = 0.$

Solution of the reduced equation

$\varphi(\omega_1, \omega_2) = \pm i\sqrt{c_2^2 + 16}\ \omega_1 + c_2\omega_2 + c_1.$

Solution of the eikonal equation

$u = \pm\sqrt{c_2x_2 - 4x_3 - i\sqrt{c_2^2 + 16}\ x_1 + c_1} - x_0.$

32. $\langle P_3 - 2X_0\rangle \oplus \langle X_1\rangle$:

    Ansatz

    $\frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u = \varphi(\omega_1, \omega_2),$

    $\omega_1 = x_2,\ \omega_2 = (x_0 + u)^2 + 4x_3.$

    Reduced equation

    $\varphi_1^2 + 16\varphi_2^2 - \omega_2 = 0.$

    Solution of the reduced equation

    $\varphi(\omega_1, \omega_2) = c_1\omega_1 + \frac{1}{6}(\omega_2 - c_1^2)^{3/2} + c_2.$

    Solution of the eikonal equation

    $\frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u = c_1x_2 +$

    $+\frac{1}{6}\left((x_0 + u)^2 + 4x_3 - c_1^2\right)^{3/2} + c_2.$

Reduction to ODEs

1. $\langle L_3\rangle \oplus \langle X_4\rangle$:

    Ansatz

    $(x_1^2 + x_2^2)^{1/2} = \varphi(\omega_1, \omega_2),$

    $\omega_1 = x_0 + u, \quad \omega_2 = x_3.$

Reduced equation

$\varphi_2^2 + 1 = 0.$

Solution of the reduced equation

$\varphi(\omega_1, \omega_2) = -i\omega_2 + f(\omega_1).$

Solution of the eikonal equation

$(x_1^2 + x_2^2)^{1/2} = -ix_3 + f(x_0 + u),$

where f is an arbitrary smooth function.

2. $\langle L_3 + \alpha X_3,\ \alpha > 0\rangle \oplus \langle X_4 \rangle$:

Ansatz

$x_3 + \alpha \arctan \dfrac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$

$\omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}.$

Reduced equation

$\omega_2^2(\varphi_2^2 + 1) + \alpha^2 = 0.$

Solution of the reduced equation

$\varphi(\omega_1, \omega_2) = -i\sqrt{\alpha^2 + \omega_2^2} + i\alpha \operatorname{arctanh} \dfrac{\alpha}{\sqrt{\alpha^2 + \omega_2^2}} +$

$+f(\omega_1).$

Solution of the eikonal equation

$x_3 + \alpha \arctan \dfrac{x_1}{x_2} = -i\sqrt{\alpha^2 + x_1^2 + x_2^2} +$

$+i\alpha \operatorname{arctanh} \dfrac{\alpha}{\alpha^2 + x_1^2 + x_2^2} + f(x_0 + u),$

where f is an arbitrary smooth function.

3. $\langle P_3 - X_1\rangle \oplus \langle X_4\rangle$:

Ansatz

$x_1 - \dfrac{x_3}{x_0+u} = \varphi(\omega_1, \omega_2),$

$\omega_1 = x_2, \quad \omega_2 = x_0 + u.$

Reduced equation

$\omega_2^2\left(\omega_2^2(\varphi_1^2+1)+1\right) = 0.$

Solutions of the reduced equation

$$\omega_2 = 0, \quad \varphi(\omega_1, \omega_2) = -i\omega_1\sqrt{\frac{1}{\omega_2^2}+1} + f(\omega_2).$$

Solutions of the eikonal equation

$$u = -x_0, \quad x_1 - \frac{x_3}{x_0+u} = -ix_2\sqrt{\frac{1}{(x_0+u)^2}+1} +$$

$+f(x_0+u),$

where f is an arbitrary smooth function.

4. $\langle P_3\rangle \oplus \langle X_4\rangle$:

Ansatz

$x_1 = \varphi(\omega_1, \omega_2),$

$\omega_1 = x_2, \quad \omega_2 = x_0 + u.$

Reduced equation

$\varphi_1^2 + 1 = 0.$

Solution of the reduced equation

$\varphi(\omega_1, \omega_2) = i\omega_1 + f(\omega_2).$

Solution of the eikonal equation

$x_1 = ix_2 + f(x_0 + u),$

where f is an arbitrary smooth function.

5. $\langle X_1 \rangle \oplus \langle X_4 \rangle$:

   Ansatz

   $x_3 = \varphi(\omega_1, \omega_2),$

   $\omega_1 = x_0 + u, \quad \omega_2 = x_2.$

   Reduced equation

   $\varphi_2^2 + 1 = 0.$

   Solution of the reduced equation

   $\varphi(\omega_1, \omega_2) = -i\omega_2 + f(\omega_1).$

   Solution of the eikonal equation

   $x_3 = -ix_2 + f(x_0 + u),$

   where f is an arbitrary smooth function.

There are no reductions

From the invariants of the remaining five nonconjugate subalgebras it is impossible to construct ansatzes, which reduce the eikonal equation. The details on this theme can be found in Chapter 2.

3.2.2 Lie Algebras of the Type A_2

In Chapter 2, we have presented six ansatzes, which are invariant with respect to two-dimensional nonconjugate subalgebras of the type A_2.

By now, we performed the symmetry reduction of the eikonal equation to differential equations with a fewer number of independent variables using those ansatzes. Some classes of invariant solutions are constructed.

Below, we present the results obtained.

Reduction to PDEs

1. $\langle -G,\ P_3 \rangle :$

   Ansatz

   $(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$

   $\omega_1 = x_1, \quad \omega_2 = x_2.$

   Reduced equation

   $(\varphi_1^2 + \varphi_2^2 - 1)\varphi^2 = 0.$

   Solutions of the reduced equation

   $\varphi(\omega_1, \omega_2) = -\sqrt{1 - c_2^2}\, \omega_1 + c_2 \omega_2 + c_1, \quad \varphi = 0.$

   Solutions of the eikonal equation

   $(x_0^2 - x_3^2 - u^2)^{1/2} = -\sqrt{1 - c_2^2} x_1 + c_2 x_2 + c_1,$

   $x_0^2 - x_3^2 - u^2 = 0.$

2. $\left\langle -G - \frac{1}{\lambda} L_3,\ P_3,\ \lambda > 0 \right\rangle :$

   Ansatz

   $(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2),$

   $\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2}.$

   Reduced equation

   $\varphi(\lambda^2 \varphi \varphi_2^2 + \varphi \omega_1^2 \varphi_1^2 - \varphi \omega_1^2 + 2\omega_1^2 \varphi_2) = 0.$

Solution of the reduced equation

$\varphi = 0.$

Solution of the eikonal equation

$x_0^2 - x_3^2 - u^2 = 0.$

3. $\left\langle -G - \dfrac{1}{\lambda} L_3,\ X_4,\ \lambda > 0 \right\rangle :$

Ansatz

$\ln(x_0 + u) + \lambda \arctan \dfrac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$

$\omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}.$

Reduced equation

$\omega_2^2(\varphi_1^2 + \varphi_2^2) + \lambda^2 = 0.$

Solution of the reduced equation

$\varphi(\omega_1, \omega_2) = i\lambda \operatorname{arctanh} \dfrac{\lambda}{\sqrt{c_1^2 \omega_2^2 + \lambda^2}} -$

$-i\sqrt{c_1^2 \omega_2^2 + \lambda^2} + c_1\omega_1 + c_2.$

Solution of the eikonal equation

$\ln(x_0 + u) = i\lambda \operatorname{arctanh} \dfrac{\lambda}{\sqrt{c_1^2(x_1^2 + x_2^2) + \lambda^2}} -$

$-i\sqrt{c_1^2(x_1^2 + x_2^2) + \lambda^2} - \lambda \arctan \dfrac{x_1}{x_2} +$

$+c_1 x_3 + c_2.$

4. $\langle -G - \alpha X_1,\ X_4,\ \alpha > 0 \rangle :$

Ansatz

$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega_1, \omega_2),$

$\omega_1 = x_2, \quad \omega_2 = x_3.$

Reduced equation

$\varphi_1^2 + \varphi_2^2 + 1 = 0.$

Solution of the reduced equation

$\varphi(\omega_1, \omega_2) = i(c_2^2 + 1)^{1/2}\omega_1 + c_2\omega_2 + c_1.$

Solution of the eikonal equation

$x_1 - \alpha \ln(x_0 + u) = i(c_2^2 + 1)^{1/2} x_2 + c_2 x_3 + c_1.$

5. $\left\langle -\frac{1}{\lambda}(L_3 + \lambda G + \alpha X_3),\ X_4,\ \alpha > 0, \lambda > 0 \right\rangle :$

Ansatz

$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2),$

$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_3 + \alpha \arctan \frac{x_1}{x_2}.$

Reduced equation

$\omega_1^2(\varphi_1^2 + \varphi_2^2) + (\alpha\varphi_2 - \lambda)^2 = 0.$

Solution of the reduced equation

$$\varphi(\omega_1, \omega_2) = -i(\alpha c_2 - \lambda) \operatorname{arctanh} \frac{\alpha c_2 - \lambda}{\sqrt{c_2^2\omega_1^2 + (\alpha c_2 - \lambda)^2}} +$$
$$+ i\sqrt{c_2^2\omega_1^2 + (\alpha c_2 - \lambda)^2} + c_2\omega_2 + c_1.$$

Solution of the eikonal equation

$$\ln(x_0 + u) = i\sqrt{c_2^2(x_1^2 + x_2^2) + (\alpha c_2 - \lambda)^2} -$$
$$- i(\alpha c_2 - \lambda) \operatorname{arctanh} \frac{\alpha c_2 - \lambda}{\sqrt{c_2^2(x_1^2 + x_2^2) + (\alpha c_2 - \lambda)^2}} +$$
$$+ (\alpha c_2 - \lambda) \arctan \frac{x_1}{x_2} + c_2 x_3 + c_1.$$

6. $\langle -G - \alpha X_1,\ P_3,\ \alpha > 0\rangle$:

Ansatz

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega_1, \omega_2),$$
$$\omega_1 = x_2, \quad \omega_2 = (x_0^2 - x_3^2 - u^2)^{1/2}.$$

Reduced equation

$$\omega_2 \left(\omega_2(\varphi_1^2 - \varphi_2^2 + 1) - 2\alpha\varphi_2\right) = 0.$$

Solutions of the reduced equation

$$\omega_2 = 0,\ \varphi(\omega_1, \omega_2) = \alpha \ln\left(2\alpha\frac{\sqrt{(c_1^2+1)\omega_2^2 + \alpha^2} + \alpha}{\omega_2^2}\right) -$$
$$-\sqrt{(c_1^2+1)\omega_2^2 + \alpha^2} + c_1\omega_1 + c_2.$$

Solutions of the eikonal equation

$$x_0^2 - x_3^2 - u^2 = 0, \quad x_1 - \alpha \ln(x_0 + u) =$$
$$= \alpha \ln\left(2\alpha\frac{\sqrt{(c_1^2+1)(x_0^2 - x_3^2 - u^2) + \alpha^2} + \alpha}{x_0^2 - x_3^2 - u^2}\right) -$$
$$-\sqrt{(c_1^2+1)(x_0^2 - x_3^2 - u^2) + \alpha^2} + c_1x_2 + c_2.$$

There are no reductions

From the invariants of the remaining one nonconjugate subalgebra it is impossible to construct ansatz, which reduces the eikonal equation. The details on this theme can be found in Chapter 2.

3.3 Classification of symmetry reductions using three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$

In this section, we present the results of the classification of symmetry reductions of the eikonal equation for all nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of dimension 3.

3.3.1 Lie Algebras of the Type $3A_1$

In Chapter 2, we have presented 25 ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $3A_1$.

By now, we performed the symmetry reduction of the eikonal equation to differential equations with a fewer number of independent variables using those ansatzes. Some classes of invariant solutions are constructed.

Below, we present the results obtained.

Reductions to algebraic equations

The invariants of five subalgebras allow us to construct the ansatzes, which reduce the eikonal equation to algebraic equations.

1. $\langle P_1 - \gamma X_3, \gamma > 0\rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta \neq 0\rangle \oplus \langle X_4\rangle :$

   Ansatz

   $x_3(x_0+u)^2 - (\gamma x_1 + x_2\delta - x_3)(x_0+u) - \gamma x_1 = \varphi(\omega),$

   $\omega = x_0 + u.$

Reduced equation

$\omega^4 + 2\omega^3 + (\gamma^2 + \delta^2 + 1)\omega^2 + 2\gamma^2\omega + \gamma^2 = 0.$

Solution of the eikonal equation

$(x_0 + u)^4 + 2(x_0 + u)^3 + (\gamma^2 + \delta^2 + 1)(x_0 + u)^2 +$

$+2\gamma^2(x_0 + u) + \gamma^2 = 0.$

2. $\langle P_1 - \gamma X_3, \gamma > 0\rangle \oplus \langle P_2 - X_2\rangle \oplus \langle X_4\rangle :$

Ansatz

$x_3(x_0 + u)^2 - (\gamma x_1 - x_3)(x_0 + u) - \gamma x_1 = \varphi(\omega),$

$\omega = x_0 + u.$

Reduced equation

$(\omega + 1)^2(\omega^2 + \gamma^2) = 0.$

Solutions of the eikonal equation

$u = -1 - x_0,\ (x_0 + u)^2 + \gamma^2 = 0.$

3. $\langle P_1\rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta > 0\rangle \oplus \langle X_4\rangle :$

Ansatz

$x_3(x_0 + u) - x_2\delta + x_3 = \varphi(\omega), \quad \omega = x_0 + u.$

Reduced equation

$\omega^2 + 2\omega + \delta^2 - 1 = 0$

Solution of the eikonal equation

$(x_0 + u)^2 + 2(x_0 + u) + \delta^2 + 1 = 0.$

As we see, the left hand sides of the Ansatzes (1)–(3) are polinomials in invariant $\omega = x_0 + u$. The reduced equations are also polinomials in ω, but with the constant coefficients. The solutions of the eikonal equation are also polinomials in variable $x_0 + u$ with the constant coefficients.

4. $\langle P_1 - X_3 \rangle \oplus \langle P_2 \rangle \oplus \langle X_4 \rangle$:

   Ansatz

   $$x_3 - \frac{x_1}{x_0 + u} = \varphi(\omega), \quad \omega = x_0 + u.$$

   Reduced equation

   $$(1 + \omega^2)\omega^2 = 0.$$

   Solutions of the reduced equation

   $$1 + \omega^2 = 0, \quad \omega = 0.$$

   Solutions of the eikonal equation

   $$1 + (x_0 + u)^2 = 0, \; u = -x_0.$$

5. $\langle P_3 - X_2 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle$:

   Ansatz

   $$x_2 - \frac{x_3}{x_0 + u} = \varphi(\omega), \quad \omega = x_0 + u.$$

   Reduced equation

   $$(\omega^2 + 1)\omega^2 = 0.$$

   Solutions of the reduced equation

   $$\omega^2 + 1 = 0, \quad \omega = 0.$$

   Solutions of the eikonal equation

   $$(x_0 + u)^2 + 1 = 0, \; u = -x_0.$$

It should be noted that subalgebras (1)–(5) belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

Reductions to linear ODEs

The invariants of six subalgebras allow us to construct the ansatzes, which reduce the eikonal equation to linear ODEs.

1. $\langle P_1 \rangle \oplus \langle P_2 \rangle \oplus \langle X_3 \rangle$:

   Ansatz

   $x_0^2 - x_1^2 - x_2^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u.$

   Reduced equation

   $\omega\varphi' - \varphi = 0.$

   Solution of the reduced equation

   $\varphi(\omega) = c_1\omega.$

   Solution of the eikonal equation

   $x_0^2 - x_1^2 - x_2^2 - u^2 = c_1(x_0 + u).$

2. $\langle P_3 \rangle \oplus \langle X_1 \rangle \oplus \langle X_2 \rangle$:

   Ansatz

   $x_0^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u.$

   Reduced equation

   $\omega\varphi' - \varphi = 0.$

   Solution of the reduced equation

   $\varphi(\omega) = c_1\omega.$

Solution of the eikonal equation

$x_0^2 - x_3^2 - u^2 = c_1(x_0 + u).$

3. $\langle P_1 \rangle \oplus \langle P_2 \rangle \oplus \langle P_3 \rangle$:

Ansatz

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u.$$

Reduced equation

$$\omega\varphi' - \varphi = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = c_1\omega.$$

Solution of the eikonal equation

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = c_1(x_0 + u).$$

Let us note, that in the cases (1)–(3) we obtained the same reduced equations.

4. $\langle P_1 \rangle \oplus \langle P_2 - X_2 \rangle \oplus \langle X_3 \rangle$:

Ansatz

$$\frac{x_0^2 - x_1^2 - u^2}{x_0 + u} - \frac{x_2^2}{x_0 + u + 1} = \varphi(\omega), \quad \omega = x_0 + u.$$

Reduced equation

$$\varphi'(\omega + 1)^4\omega^4 = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = c_1, \quad \omega + 1 = 0, \quad \omega = 0.$$

Solutions of the eikonal equation

$$\frac{x_0^2 - x_1^2 - u^2}{x_0 + u} - \frac{x_2^2}{x_0 + u + 1} = c_1,$$

$$u = -1 - x_0, \quad u = -x_0.$$

5. $\langle P_1 \rangle \oplus \langle P_2 - \alpha X_2, \alpha > 0 \rangle \oplus \langle P_3 - \gamma X_3, \gamma \neq 0 \rangle :$

Ansatz

$$2u + \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} + \frac{x_3^2}{x_0 + u + \gamma} = \varphi(\omega),$$

$$\omega = x_0 + u.$$

Reduced equation

$$\omega^4 (\omega + \gamma)^4 (\omega + \alpha)^4 (\varphi' - 1) = 0.$$

Solutions of the reduced equation

$$\omega = 0, \quad \omega + \gamma = 0, \quad \omega + \alpha = 0, \quad \varphi(\omega) = \omega + c.$$

Solutions of the eikonal equation

$$u = -x_0, \quad u = -x_0 - \gamma, \quad u = -x_0 - \alpha,$$

$$2u + \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} + \frac{x_3^2}{x_0 + u + \gamma} = x_0 + u + c.$$

6. $\langle P_1 \rangle \oplus \langle P_2 - \alpha X_2, \alpha > 0 \rangle \oplus \langle P_3 \rangle :$

Ansatz

$$2u + \frac{x_1^2 + x_3^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} = \varphi(\omega), \quad \omega = x_0 + u.$$

The reduced equation

$$\omega^4 (\omega + \alpha)^4 (\varphi' - 1) = 0.$$

Solutions of the reduced equation

$\omega = 0, \quad \omega + \alpha = 0, \quad \varphi(\omega) = \omega + c.$

Solutions on the eikonal equation

$u = -x_0, \quad u = -x_0 - \alpha,$

$$2u + \frac{x_1^2 + x_3^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} = x_0 + u + c.$$

It should be noted that subalgebras (1)–(6) belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

Reductions to equations, which can be splitted on two linear ODEs

Taking into account the invariants of nine nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to those, which could be splitted on two linear ODEs.

1. $\langle X_1 \rangle \oplus \langle X_2 \rangle \oplus \langle X_4 \rangle$:

   Ansatz

   $x_0 + u = \varphi(\omega), \quad \omega = x_3.$

   Reduced equation

   $(\varphi')^2 = 0.$

   Solution of the reduced equation

   $\varphi(\omega) = c_1.$

   Solution of the eikonal equation

   $u = c_1 - x_0.$

2. $\langle P_1 \rangle \oplus \langle P_2 - X_2 \rangle \oplus \langle X_4 \rangle$:

Ansatz

$x_0 + u = \varphi(\omega), \quad \omega = x_3.$

Reduced equation

$(\varphi')^2 = 0.$

Solution of the reduced equation

$\varphi(\omega) = c_1.$

Solution of the eikonal equation

$u = c_1 - x_0.$

3. $\langle L_3 \rangle \oplus \langle X_3 \rangle \oplus \langle X_4 \rangle$:

Ansatz

$x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$

Reduced equation

$(\varphi')^2 = 0.$

Solution of the reduced equation

$\varphi(\omega) = c_1.$

Solution of the eikonal equation

$u = c_1 - x_0.$

4. $\langle P_3 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle$:

Ansatz

$x_0 + u = \varphi(\omega), \quad \omega = x_2.$

Reduced equation

$(\varphi')^2 = 0.$

Solution of the reduced equation

$\varphi(\omega) = c_1.$

Solution of the eikonal equation

$u = c_1 - x_0.$

5. $\langle L_3 \rangle \oplus \langle P_3 \rangle \oplus \langle X_4 \rangle$:

   Ansatz

   $x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$

   Reduced equation

   $(\varphi')^2 = 0.$

   Solution of the reduced equation

   $\varphi(\omega) = c_1.$

   Solution of the eikonal equation

   $u = c_1 - x_0.$

6. $\langle P_1 \rangle \oplus \langle P_2 \rangle \oplus \langle X_4 \rangle$:

   Ansatz

   $x_0 + u = \varphi(\omega), \quad \omega = x_3.$

   Reduced equation

   $(\varphi')^2 = 0.$

   Solution of the reduced equation

   $\varphi(\omega) = c_1.$

Solution of the eikonal equation

$u = c_1 - x_0.$

Let us note that, in the cases (1)–(6), we obtained the same reduced equations. The solutions of the eikonal equation are also the same.

7. $\langle G \rangle \oplus \langle X_2 \rangle \oplus \langle X_1 \rangle$:

   Ansatz

   $(x_0^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3.$

   Reduced equation

   $(\varphi' - 1)(\varphi' + 1) = 0.$

   Solutions of the reduced equation

   $\varphi(\omega) = \varepsilon\omega + c_1, \ \varepsilon = \pm 1.$

   Solutions of the eikonal equation

   $(x_0^2 - u^2)^{1/2} = \varepsilon x_3 + c_1, \ \varepsilon = \pm 1.$

8. $\langle G \rangle \oplus \langle L_3 \rangle \oplus \langle X_3 \rangle$:

   Ansatz

   $(x_0^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$

   Reduced equation

   $(\varphi' - 1)(\varphi' + 1) = 0.$

   Solutions of the reduced equation

   $\varphi(\omega) = \varepsilon\omega + c_1, \ \varepsilon = \pm 1.$

Solutions of the eikonal equation

$(x_0^2 - u^2)^{1/2} = \varepsilon(x_1^2 + x_2^2)^{1/2} + c_1,\ \varepsilon = \pm 1.$

9. $\langle P_3 - 2X_0\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle :$

   Ansatz

   $$\frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u = \varphi(\omega),$$
   $$\omega = (x_0 + u)^2 + 4x_3.$$

   Reduced equation

   $$16(\varphi')^2 - \omega = 0.$$

   Solutions of the reduced equation

   $$\varphi(\omega) = \frac{\varepsilon}{6}\omega^{3/2} + c_1,\ \varepsilon = \pm 1.$$

   Solutions of the eikonal equation

   $$\frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u =$$
   $$= \frac{\varepsilon}{6}\left((x_0 + u)^2 + 4x_3\right)^{3/2} + c_1,\ \varepsilon = \pm 1.$$

Reductions to nonlinear ODEs

From the invariants of five nonconjugate subalgebras we constructed the ansatzes, which reduced the eikonal equation to nonlinear ODEs.

1. $\langle G + \alpha X_3, \alpha > 0\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle :$

   Ansatz

   $$x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_0^2 - u^2.$$

Reduced equation

$4\omega(\varphi')^2 + 4\alpha\varphi' - 1 = 0.$

Solutions of the reduced equation

$\varphi(\omega) = \varepsilon(\alpha^2+\omega)^{1/2} - \varepsilon\alpha\mathrm{arctanh}\dfrac{(\alpha^2+\omega)^{1/2}}{\alpha} - \dfrac{\alpha}{2}\ln(\omega) +$

$+c_1,\ \varepsilon = \pm 1.$

Solutions of the eikonal equation

$x_3 - \alpha\ln(x_0+u) = \varepsilon(\alpha^2 + x_0^2 - u^2)^{1/2} - \dfrac{\alpha}{2}\ln(x_0^2 - u^2) -$

$-\varepsilon\alpha\mathrm{arctanh}\dfrac{(\alpha^2 + x_0^2 - u^2)^{1/2}}{\alpha} + c_1,\ \varepsilon = \pm 1.$

2. $\langle L_3\rangle \oplus \langle P_3 + C_3\rangle \oplus \langle X_0 + X_4\rangle$:

Ansatz

$(x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$

Reduced equation

$(1 + (\varphi')^2)\varphi^2 = 0.$

Solutions of the reduced equation

$\varphi(\omega) = i\varepsilon\omega + c_1, \varepsilon = \pm 1;\ \varphi = 0.$

Solutions of the eikonal equation

$(x_3^2 + u^2)^{1/2} = i\varepsilon(x_1^2 + x_2^2)^{1/2} + c_1, \varepsilon = \pm 1;\ x_3^2 + u^2 = 0.$

3. $\langle L_3 + \alpha(X_0 + X_4), \alpha > 0\rangle \oplus \langle X_3\rangle \oplus \langle X_4\rangle$:

Ansatz

$x_0 + u + \alpha\arctan\dfrac{x_2}{x_1} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$

Reduced equation

$\omega^2(\varphi')^2 + \alpha^2 = 0.$

Solutions of the reduced equation

$\varphi(\omega) = i\varepsilon\alpha\ln(\omega) + c_1,\ \varepsilon = \pm 1.$

Solutions of the eikonal equation

$x_0 + u + \alpha\arctan\dfrac{x_2}{x_1} = i\dfrac{\varepsilon\alpha}{2}\ln(x_1^2 + x_2^2) + c_1,\ \varepsilon = \pm 1.$

4. $\langle P_3 - 2X_0\rangle \oplus \langle X_1\rangle \oplus \langle X_4\rangle$:

   Ansatz

   $(x_0 + u)^2 + 4x_3 = \varphi(\omega), \quad \omega = x_2.$

   Reduced equation

   $(\varphi')^2 + 16 = 0.$

   Solutions of the reduced equation

   $\varphi(\omega) = 4i\varepsilon\omega + c_1,\ \varepsilon = \pm 1.$

   Solutions of the eikonal equation

   $(x_0 + u)^2 + 4x_3 = 4i\varepsilon x_2 + c_1,\ \varepsilon = \pm 1.$

5. $\langle L_3\rangle \oplus \langle -P_3 + 2X_0\rangle \oplus \langle 2X_4\rangle$:

   Ansatz

   $(x_0 + u)^2 + 4x_3 = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$

   Reduced equation

   $(\varphi')^2 + 16 = 0.$

Solutions of the reduced equation

$\varphi(\omega) = 4i\varepsilon\omega + c_1, \varepsilon = \pm 1.$

Solutions of the eikonal equation

$(x_0 + u)^2 + 4x_3 = 4i\varepsilon(x_1^2 + x_2^2)^{1/2} + c_1,\ \varepsilon = \pm 1.$

There are no reductions

From the invariants of the remaining six nonconjugate subalgebras it is impossible to construct the ansatzes, which reduce the eikonal equation. The details on this theme can be found in Chapter 2.

3.3.2 Lie Algebras of the Type $A_2 \oplus A_1$

In Chapter 2, we have presented seven ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_2 \oplus A_1$.

By now, we performed the symmetry reduction of the eikonal equation to differential equations with a fewer number of independent variables using those ansatzes. Some classes of invariant solutions are constructed.

Below, we present the results obtained.

Reductions to equations, which can be splitted on two linear ODEs

Taking into account the invariants of two nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to those, which could be splitted on two linear ODEs.

1. $\langle -G,\ P_3\rangle \oplus \langle X_1\rangle$:

   Ansatz

   $(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_2.$

Reduced equation

$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0.$

Solutions of the reduced equation

$\varphi(\omega) = \varepsilon\omega + c_1, \quad \varphi = 0, \quad \varepsilon = \pm 1.$

Solutions of the eikonal equation

$(x_0^2 - x_3^2 - u^2)^{1/2} = \varepsilon x_2 + c_1, \quad x_0^2 - x_3^2 - u^2 = 0, \quad \varepsilon = \pm 1.$

2. $\langle -G,\ P_3\rangle \oplus \langle L_3\rangle :$

   Ansatz

   $(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$

   Reduced equation

   $(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0.$

   Solutions of the reduced equation

   $\varphi(\omega) = \varepsilon\omega + c_1, \quad \varphi = 0, \quad \varepsilon = \pm 1.$

   Solutions of the eikonal equation

   $(x_0^2 - x_3^2 - u^2)^{1/2} = \varepsilon(x_1^2 + x_2^2)^{1/2} + c_1,$

   $x_0^2 - x_3^2 - u^2 = 0, \quad \varepsilon = \pm 1.$

It should be noted that subalgebras (1) and (2) do not belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

Reductions to nonlinear ODEs

From the invariants of five nonconjugate subalgebras we constructed the ansatzes, which reduced the eikonal equation to nonlinear ODEs.

1. $\langle -\left(G+\alpha X_2\right),\ P_3, \alpha>0\rangle \oplus \langle X_1\rangle$:

Ansatz

$x_2-\alpha \ln(x_0+u)=\varphi(\omega), \quad \omega=(x_0^2-x_3^2-u^2)^{1/2}.$

Reduced equation

$\omega\left(\omega(\varphi')^2+2\alpha\varphi'-\omega\right)=0.$

Solutions of the reduced equation

$\omega=0,\ \varphi(\omega)=\varepsilon(\alpha^2+\omega^2)^{1/2}-\alpha\ln(\omega)-$
$-\varepsilon\alpha \text{arctanh}\dfrac{\alpha}{(\alpha^2+\omega^2)^{1/2}}+c_1,\ \varepsilon=\pm 1.$

Solutions of the eikonal equation

$x_0^2-x_3^2-u^2=0, \quad x_2-\alpha\ln(x_0+u)=$
$=\varepsilon(x_0^2-x_3^2-u^2+\alpha^2)^{1/2}-\dfrac{\alpha}{2}\ln(x_0^2-x_3^2-u^2)-$
$-\varepsilon\alpha \text{arctanh}\dfrac{\alpha}{(x_0^2-x_3^2-u^2+\alpha^2)^{1/2}}+c_1, \quad \varepsilon=\pm 1.$

2. $\left\langle -\dfrac{1}{\lambda}L_3-G,\ 2X_4, \lambda>0\right\rangle \oplus \langle X_3\rangle$:

Ansatz

$\ln(x_0+u)+\lambda \arctan\dfrac{x_1}{x_2}=\varphi(\omega), \quad \omega=(x_1^2+x_2^2)^{1/2}.$

Reduced equation

$\omega^2(\varphi')^2+\lambda^2=0.$

Solutions of the reduced equation

$\varphi(\omega)=i\varepsilon\lambda\ln(\omega)+c_1, \quad \varepsilon=\pm 1.$

Solutions of the eikonal equation

$$\ln(x_0+u)+\lambda \arctan\frac{x_1}{x_2} = i\frac{\varepsilon\lambda}{2}\ln(x_1^2+x_2^2)+c_1, \quad \varepsilon = \pm 1.$$

3. $\langle -(G+\alpha X_2),\ X_4, \alpha > 0\rangle \oplus \langle X_1\rangle :$

Ansatz

$$x_2 - \alpha\ln(x_0+u) = \varphi(\omega), \quad \omega = x_3.$$

Reduced equation

$$(\varphi')^2 + 1 = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = i\varepsilon\omega + c_1, \quad \varepsilon = \pm 1.$$

Solutions of the eikonal equation

$$x_2 - \alpha\ln(x_0+u) = i\varepsilon x_3 + c_1, \quad \varepsilon = \pm 1.$$

4. $\langle -(G+\alpha X_3),\ X_4,\ \alpha > 0\rangle \oplus \langle L_3 + \beta X_3,\ \beta > 0\rangle :$

Ansatz

$$x_3 - \alpha\ln(x_0+u) + \beta\arctan\frac{x_1}{x_2} = \varphi(\omega),$$

$$\omega = (x_1^2+x_2^2)^{1/2}.$$

Reduced equation

$$\omega^2(\varphi')^2 + \omega^2 + \beta^2 = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = i\varepsilon(\omega^2+\beta^2)^{1/2} - i\varepsilon\beta \operatorname{arctanh}\frac{\beta}{(\omega^2+\beta^2)^{1/2}} + c_1,$$

$$\varepsilon = \pm 1.$$

Solutions of the eikonal equation

$$x_3 - \alpha \ln(x_0 + u) + \beta \arctan \frac{x_1}{x_2} =$$

$$= -i\varepsilon\beta \text{arctanh} \frac{\beta}{(x_1^2 + x_2^2 + \beta^2)^{1/2}} +$$

$$+ i\varepsilon(x_1^2 + x_2^2 + \beta^2)^{1/2} + c_1, \quad \varepsilon = \pm 1.$$

5. $\langle -(G + \alpha X_3)\,,\ X_4, \alpha > 0\rangle \oplus \langle L_3\rangle$:

   Ansatz

   $$x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

   Reduced equation

   $$(\varphi')^2 + 1 = 0.$$

   Solutions of the reduced equation

   $$\varphi(\omega) = i\varepsilon\omega + c_1, \quad \varepsilon = \pm 1.$$

   Solutions of the eikonal equation

   $$x_3 - \alpha \ln(x_0 + u) = i\varepsilon(x_1^2 + x_2^2)^{1/2} + c_1, \quad \varepsilon = \pm 1.$$

It should be noted that subalgebras (1)–(5) do not belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

There are no reductions

From the invariants of the remaining three nonconjugate subalgebras it is impossible to construct the ansatzes, which reduce the eikonal equation. The details on this theme can be found in Chapter 2.

3.3.3 Lie Algebras of the Type $A_{3,1}$

In Chapter 2, we have presented 16 ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,1}$.

By now, we performed the symmetry reduction of the eikonal equation to differential equations with a fewer number of independent variables using those ansatzes. Some classes of invariant solutions are constructed.

Below, we present the results obtained.

Reductions to algebraic equations

Taking into account the invariants of seven nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to algebraic equations.

1. $\langle 4X_4,\ P_1 - X_2 - \gamma X_3,\ P_2 + X_1 - \mu X_2 - \delta X_3,$
   $\gamma > 0, \delta \neq 0, \mu > 0\rangle :$

   Ansatz

   $$x_3(x_0+u)^2 - (\gamma x_1 + x_2\delta - \mu x_3)(x_0+u) +$$
   $$+(\delta - \gamma\mu)x_1 - x_2\gamma + x_3 = \varphi(\omega), \quad \omega = x_0 + u.$$

   Reduced equation

   $$\omega^4 + 2\mu\omega^3 + (\gamma^2 + \mu^2 + \delta^2 + 2)\omega^2 + 2\mu(\gamma^2+1)\omega +$$
   $$+(\gamma\mu - \delta)^2 + \gamma^2 + 1 = 0.$$

   Solution of the eikonal equation

   $$(x_0+u)^4 + 2\mu(x_0+u)^3 + (\gamma^2+\mu^2+\delta^2+2)(x_0+u)^2 +$$
   $$+2\mu(\gamma^2+1)(x_0+u) + (\gamma\mu-\delta)^2 + \gamma^2 + 1 = 0.$$

2. $\langle 4X_4,\ P_1 - X_2 - \gamma X_3,\ P_2 + X_1 - \mu X_2, \gamma > 0, \mu > 0\rangle :$

Ansatz
$x_3(x_0+u)^2 - (\gamma x_1 - \mu x_3)(x_0+u) - \gamma\mu x_1 - x_2\gamma + x_3 =$
$= \varphi(\omega), \quad \omega = x_0 + u.$

Reduced equation
$\omega^4 + 2\mu\omega^3 + (\gamma^2+\mu^2+2)\omega^2 + 2\mu(\gamma^2+1)\omega + \gamma^2(\mu^2+1) + 1 =$
$= 0.$

Solution of the eikonal equation
$(x_0+u)^4 + 2\mu(x_0+u)^3 + (\gamma^2+\mu^2+2)(x_0+u)^2 +$
$+2\mu(\gamma^2+1)(x_0+u) + \gamma^2(\mu^2+1) + 1 = 0.$

3. $\langle 4X_4,\ P_1 - X_2,\ P_2 + X_1 - \mu X_2 - \delta X_3, \delta > 0, \mu \neq 0\rangle :$

Ansatz
$x_3(x_0+u)^2 - (x_2\delta - \mu x_3)(x_0+u) + \delta x_1 + x_3 = \varphi(\omega),$
$\omega = x_0 + u.$

Reduced equation
$\omega^4 + 2\mu\omega^3 + (\delta^2+\mu^2+2)\omega^2 + 2\mu\omega + \delta^2 + 1 = 0.$

Solution of the eikonal equation
$(x_0+u)^4 + 2\mu(x_0+u)^3 + (\delta^2+\mu^2+2)(x_0+u)^2 +$
$+2\mu(x_0+u) + \delta^2 + 1 = 0.$

4. $\langle 4X_4,\ P_1 - X_2,\ P_2 + X_1 - \delta X_3, \delta > 0\rangle$:

Ansatz

$x_3(x_0+u)^2 - x_2\delta(x_0+u) + \delta x_1 + x_3 = \varphi(\omega), \quad \omega = x_0+u.$

Reduced equation

$(\omega^2 + 1)(\omega^2 + \delta^2 + 1) = 0.$

Solutions of the eikonal equation

$(x_0 + u)^2 + 1 = 0,\ (x_0 + u)^2 + \delta^2 + 1 = 0.$

5. $\langle 4X_4,\ P_1 - X_2 - \beta X_3,\ P_2 + X_1, \beta > 0\rangle$:

Ansatz

$x_3(x_0+u)^2 - \beta x_1(x_0+u) - \beta x_2 + x_3 = \varphi(\omega), \quad \omega = x_0+u.$

Reduced equation

$(\omega^2 + 1)(\omega^2 + \beta^2 + 1) = 0.$

Solutions of the eikonal equation

$(x_0 + u)^2 + 1 = 0,\ (x_0 + u)^2 + \beta^2 + 1 = 0.$

As we see, the left hand sides of the Ansatzes (1)–(5) are polinomials in invariant $\omega = x_0 + u$. The reduced equations are also polinomials in variable ω, but with the constant coefficients. The solutions of the eikonal equation are also polinomials in variable $x_0 + u$ with the constant coefficients.

6. $\langle 4X_4,\ P_1 - X_2,\ P_2 + X_1 - \mu X_2, \mu \neq 0\rangle$:

   Ansatz

   $x_3(x_0+u)^2 + \mu x_3(x_0+u) + x_3 = \varphi(\omega), \quad \omega = x_0 + u.$

   Reduced equation

   $(\omega^2 + \mu\omega + 1)^2 = 0.$

   Solution of the eikonal equation

   $u = -\frac{1}{2}\left(\mu + (\mu^2-4)^{1/2}\right) - x_0.$

7. $\langle 2\mu X_4,\ P_3 - X_2,\ X_1 + \mu X_3, \mu > 0\rangle$:

   Ansatz

   $x_2 - \frac{x_3 - \mu x_1}{x_0 + u} = \varphi(\omega), \quad \omega = x_0 + u.$

   Reduced equation

   $\omega^2(\omega^2 + \mu^2 + 1) = 0.$

   Solutions of the reduced equation

   $\omega = 0, \quad \omega^2 + \mu^2 + 1 = 0.$

   Solutions of the eikonal equation

   $u = -x_0,\ (x_0+u)^2 + \mu^2 + 1 = 0.$

It should be noted that subalgebras (1)–(7) belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

Reductions to equations, which can be splitted on two linear ODEs

Taking into account the invariants of five nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to those, which could be splitted on two linear ODEs.

1. $\langle 2\mu X_4,\ P_3,\ X_1 + \mu X_3, \mu > 0\rangle$:

   Ansatz

   $x_0 + u = \varphi(\omega), \quad \omega = x_2.$

   Reduced equation

   $(\varphi')^2 = 0.$

   Solution of the reduced equation

   $\varphi(\omega) = c_1.$

   Solution of the eikonal equation

   $u = c_1 - x_0.$

2. $\langle 2X_4,\ P_3 - L_3,\ X_3\rangle$:

   Ansatz

   $x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$

   Reduced equation

   $(\varphi')^2 = 0.$

   Solution of the reduced equation

   $\varphi(\omega) = c_1.$

   Solution of the eikonal equation

   $u = c_1 - x_0.$

3. $\langle 2X_4,\ P_3 - X_1,\ X_3\rangle$:

Ansatz

$x_0 + u = \varphi(\omega), \quad \omega = x_2.$

Reduced equation

$(\varphi')^2 = 0.$

Solution of the reduced equation

$\varphi(\omega) = c_1.$

Solution of the eikonal equation

$u = c_1 - x_0.$

4. $\langle -2\alpha X_4,\ L_3 + \alpha X_3,\ P_3, \alpha > 0\rangle$:

Ansatz

$x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$

Reduced equation

$(\varphi')^2 = 0.$

Solution of the reduced equation

$\varphi(\omega) = c_1.$

Solution of the eikonal equation

$u = c_1 - x_0.$

5. $\langle 4X_4,\ P_1 - X_2,\ P_2 + X_1\rangle$:

Ansatz

$x_0 + u = \varphi(\omega), \quad \omega = x_3.$

Reduced equation

$(\varphi')^2 = 0.$

Solution of the reduced equation

$\varphi(\omega) = c_1.$

Solution of the eikonal equation

$u = c_1 - x_0.$

Let us note that, in the cases (1)–(5), we obtained the same reduced equation. The solutions of the eikonal equation are also the same.

It should be noted that subalgebras (1)–(5) belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

Reductions to nonlinear ODEs

From the invariants of four nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to nonlinear ODEs.

1. $\langle 2\mu X_4,\ P_3 - 2X_0,\ X_1 + \mu X_3,\ \mu > 0\rangle :$

Ansatz

$(x_0 + u)^2 + 4x_3 - 4\mu x_1 = \varphi(\omega), \quad \omega = x_2.$

Reduced equation

$(\varphi')^2 + 16(\mu^2 + 1) = 0.$

Solutions of the reduced equation

$\varphi(\omega) = 4i\varepsilon(\mu^2 + 1)^{1/2}\omega + c_1,\ \varepsilon = \pm 1.$

Solutions of the eikonal equation

$u = 2\left(i\varepsilon x_2\sqrt{\mu^2 + 1} + \mu x_1 - x_3 + c_1\right)^{1/2} - x_0,\ \varepsilon = \pm 1.$

2. $\langle 2X_4,\ P_3 - L_3 - 2\alpha X_0,\ X_3, \alpha > 0\rangle$:

Ansatz

$$2\alpha \arctan \frac{x_1}{x_2} - x_0 - u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$\omega^2(\varphi')^2 + 4\alpha^2 = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = 2i\varepsilon\alpha \ln(\omega) + c_1,\ \varepsilon = \pm 1.$$

Solutions of the eikonal equation

$$u = 2\alpha \arctan \frac{x_1}{x_2} + i\varepsilon\alpha \ln(x_1^2 + x_2^2) - x_0 + c_1,\ \varepsilon = \pm 1.$$

3. $\langle -2\beta X_4,\ L_3 + \beta X_3,\ P_3 - 2X_0, \beta > 0\rangle$:

Ansatz

$$\beta \arctan \frac{x_1}{x_2} + \frac{1}{4}(x_0 + u)^2 + x_3 = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$\omega^2(\varphi')^2 + \omega^2 + \beta^2 = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = i\varepsilon\sqrt{\omega^2 + \beta^2} - i\varepsilon\beta \text{arctanh} \frac{\beta}{\sqrt{\omega^2 + \beta^2}} + c_1,$$

$\varepsilon = \pm 1.$

Solutions of the eikonal equation

$$\beta \arctan \frac{x_1}{x_2} + \frac{1}{4}(x_0 + u)^2 = i\varepsilon\sqrt{x_1^2 + x_2^2 + \beta^2} -$$

$$-i\varepsilon\beta \operatorname{arctanh}\frac{\beta}{\sqrt{x_1^2+x_2^2+\beta^2}} - x_3 + c_1,\ \varepsilon = \pm 1.$$

4. $\langle 2X_4,\ P_3,\ X_3\rangle$:

   Ansatz

   $x_2 = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 + u,\ \omega_2 = x_1.$

   Reduced equation

   $\varphi_2^2 + 1 = 0.$

   Solution of the reduced equation

   $\varphi(\omega_1, \omega_2) = i\omega_2 + f(\omega_1).$

   Solution of the eikonal equation

   $x_2 = ix_1 + f(x_0 + u),$

   where f is an arbitrary smooth function.

It should be noted that subalgebras (1)–(4) belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

There are no reductions

From the invariants of the remaining one nonconjugate subalgebra it is impossible to construct the ansatz, which reduces the eikonal equation. The details on this theme can be found in Chapter 2.

3.3.4 Lie Algebras of the Type $A_{3,2}$

In Chapter 2, we have presented two ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,2}$.

By now, we performed the symmetry reduction of the eikonal equation to differential equations with a fewer number of independent variables using those ansatzes. Some classes of invariant solutions are constructed.

Below, we present the results obtained.

Reductions to nonlinear ODEs

From the invariants of two nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to nonlinear ODEs.

1. $\langle 2\beta X_4,\ P_3,\ G+\alpha X_1+\beta X_3, \alpha>0, \beta>0\rangle$:

   Ansatz

   $x_1-\alpha\ln(x_0+u)=\varphi(\omega), \quad \omega=x_2.$

   Reduced equation

   $(\varphi')^2+1=0.$

   Solutions of the reduced equation

   $\varphi(\omega)=i\varepsilon\omega+c_1,\ \varepsilon=\pm 1.$

   Solutions of the eikonal equation

   $x_1-\alpha\ln(x_0+u)=i\varepsilon x_2+c_1,\ \varepsilon=\pm 1.$

2. $\left\langle 2\alpha X_4,\ \lambda P_3,\ \dfrac{1}{\lambda}L_3+G+\dfrac{\alpha}{\lambda}X_3, \alpha>0, \lambda>0\right\rangle$:

   Ansatz

   $\ln(x_0+u)+\lambda\arctan\dfrac{x_1}{x_2}=\varphi(\omega), \quad \omega=(x_1^2+x_2^2)^{1/2}.$

   Reduced equation

   $\omega^2(\varphi')^2+\lambda^2=0.$

Solutions of the reduced equation

$$\varphi(\omega) = i\varepsilon\lambda \ln(\omega) + c_1, \ \varepsilon = \pm 1.$$

Solutions of the eikonal equation

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = i\varepsilon \frac{\lambda}{2} \ln(x_1^2 + x_2^2) + c_1, \ \varepsilon = \pm 1.$$

It should be noted that subalgebras (1) and (2) do not belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

There are no reductions

From the invariants of the remaining one nonconjugate subalgebra it is impossible to construct the ansatz, which reduces the eikonal equation. The details on this theme can be found in Chapter 2.

3.3.5 Lie Algebras of the Type $A_{3,3}$

In Chapter 2, we have presented four ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,3}$.

By now, we performed the symmetry reduction of the eikonal equation to differential equations with a fewer number of independent variables using those ansatzes. Some classes of invariant solutions are constructed.

Below, we present the results obtained.

Reductions to equations, which can be splitted on two linear ODEs

Taking into account the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to that, which could be splitted on two linear ODEs.

$\langle P_1,\ P_2,\ G\rangle :$

Ansatz
$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3.$

Reduced equation
$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0.$

Solutions of the reduced equation
$\varphi(\omega) = \varepsilon\omega + c,\ \varepsilon = \pm 1, \quad \varphi = 0.$

Solutions of the eikonal equation
$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varepsilon x_3 + c_1,\ \varepsilon = \pm 1,$
$x_0^2 - x_1^2 - x_2^2 - u^2 = 0.$

It should be noted that subalgebra do not belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

Reductions to nonlinear ODEs

From the invariants of three subalgebras, we constructed the ansatzes, which reduced the eikonal equation to nonlinear ODEs.

1. $\langle P_1,\ P_2,\ G + \alpha X_3, \alpha > 0\rangle :$

   Ansatz

   $x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_0^2 - x_1^2 - x_2^2 - u^2.$

   Reduced equation

   $4\omega\,(\varphi')^2 + 4\alpha\varphi' - 1 = 0.$

Solutions of the reduced equation

$$\varphi(\omega) = \varepsilon(\alpha^2 + \omega)^{1/2} - i\varepsilon\alpha \arctan \frac{(\alpha^2 + \omega)^{1/2}}{i\alpha} -$$
$$-\frac{\alpha}{2} \ln \omega + c, \quad \varepsilon = \pm 1.$$

Solutions of the eikonal equation

$$x_3 - \alpha \ln(x_0 + u) = \varepsilon(x_0^2 - x_1^2 - x_2^2 - u^2 + \alpha^2)^{1/2} -$$
$$-i\varepsilon\alpha \arctan \frac{(x_0^2 - x_1^2 - x_2^2 - u^2 + \alpha^2)^{1/2}}{i\alpha} -$$
$$-\frac{\alpha}{2} \ln(x_0^2 - x_1^2 - x_2^2 - u^2) + c, \quad \varepsilon = \pm 1.$$

2. $\left\langle P_3,\ X_4,\ \frac{1}{\lambda} L_3 + G, \lambda > 0 \right\rangle$:

Ansatz
$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation
$$\omega^2(\varphi')^2 + \lambda^2 = 0.$$

Solutions of the reduced equation
$$\varphi(\omega) = i\varepsilon\lambda \ln(\omega) + c,\ \varepsilon = \pm 1.$$

Solutions of the eikonal equation
$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = i\varepsilon \frac{\lambda}{2} \ln(x_1^2 + x_2^2) + c,\ \varepsilon = \pm 1.$$

3. $\langle P_3,\ X_4,\ G + \alpha X_1, \alpha > 0 \rangle$:

Ansatz

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_2.$$

Reduced equation

$$(\varphi')^2 + 1 = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = i\varepsilon\omega + c, \quad \varepsilon = \pm 1.$$

Solutions of the eikonal equation

$$x_1 - \alpha \ln(x_0 + u) = i\varepsilon x_2 + c_1, \quad \varepsilon = \pm 1.$$

It should be noted that subalgebras (1)–(3) do not belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

There are no reduction

From the invariants of the remaining one nonconjugate subalgebra it is impossible to construct the ansatz, which reduces the eikonal equation. The details on this theme can be found in Chapter 2.

3.3.6 Lie Algebras of the Type $A_{3,4}$

From the invariants of all four nonconjugate subalgebras it is impossible to construct the ansatzes, which reduce the eikonal equation. More details on this theme can be found in Chapter 2.

It should be noted that those subalgebras do not belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

3.3.7 Lie Algebras of the Type $A^a_{3,5}$

The Lie algebra of the group $P(1,4)$ contains no nonconjugate subalgebras of the type $A^a_{3,5}$.

3.3.8 Lie Algebras of the Type $A_{3,6}$

In Chapter 2, we have presented 16 ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,6}$.

By now, we performed the symmetry reduction of the eikonal equation to differential equations with a fewer number of independent variables using those ansatzes. Some classes of invariant solutions are constructed.

Below, we present the results obtained.

Reductions to linear ODEs

The invariants of four subalgebras allow us to construct the ansatzes, which reduce the eikonal equation to linear ODEs.

1. $\langle P_1 - X_1,\ P_2 - X_2,\ -P_3 + L_3 \rangle$:

   Ansatz

   $$\frac{x_1^2 + x_2^2}{x_0 + u + 1} + \frac{x_3^2}{x_0 + u} + 2u = \varphi(\omega), \quad \omega = x_0 + u.$$

   Reduced equation

   $$\omega^4(\omega + 1)^4(\varphi' - 1) = 0.$$

   Solutions of the reduced equation

   $$\omega + 1 = 0, \quad \omega = 0, \quad \varphi(\omega) = \omega + c_1.$$

Solutions of the eikonal equation

$$u = -1 - x_0,\ u = -x_0, \quad \frac{x_1^2 + x_2^2}{x_0 + u + 1} + \frac{x_3^2}{x_0 + u} + 2u =$$
$$= x_0 + u + c_1.$$

2. $\langle P_1,\ -P_2,\ -(L_3 + \alpha X_3)\,, \alpha > 0\rangle :$

Ansatz

$$x_0^2 - x_1^2 - x_2^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u.$$

Reduced equation

$$\omega\varphi' - \varphi = 0$$

Solution of the reduced equation

$$\varphi(\omega) = c\omega.$$

Solution of the eikonal equation

$$x_0^2 - x_1^2 - x_2^2 - u^2 = c(x_0 + u).$$

3. $\langle X_1,\ -X_2,\ P_3 - L_3\rangle :$

Ansatz

$$x_0^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u.$$

Reduced equation

$$\omega\varphi' - \varphi = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = c_1\omega.$$

Solution of the eikonal equation

$$x_0^2 - x_3^2 - u^2 = c_1(x_0 + u).$$

4. $\langle P_1,\ P_2,\ -P_3 + L_3\rangle$:

Ansatz

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u.$$

Reduced equation

$$\omega\varphi' - \varphi = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = c_1\omega.$$

Solution of the eikonal equation

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = c_1(x_0 + u).$$

It should be noted that subalgebras (1)–(4) belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

Reductions to equations, which can be splitted on two linear ODEs

Taking into account the invariants of seven nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to those, which could be splitted on two linear ODEs.

1. $\langle X_1,\ -X_2,\ -(L_3+2X_4)\rangle$:

Ansatz

$x_0 + u = \varphi(\omega), \quad \omega = x_3.$

Reduced equation

$(\varphi')^2 = 0.$

Solution of the reduced equation

$\varphi(\omega) = c_1.$

Solution of the eikonal equation

$u = c_1 - x_0.$

2. $\langle P_1,\ P_2,\ L_3+2X_4\rangle$:

Ansatz

$x_0 + u = \varphi(\omega), \quad \omega = x_3.$

Reduced equation

$(\varphi')^2 = 0.$

Solution of the reduced equation

$\varphi(\omega) = c_1.$

Solution of the eikonal equation

$u = c_1 - x_0.$

3. $\left\langle X_1,\ X_2,\ L_3+\frac{1}{2}(P_3+C_3)\right\rangle$:

Ansatz

$(x_3^2+u^2)^{1/2} = \varphi(\omega), \quad \omega = x_0.$

Reduced equation

$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0.$

Solutions of the reduced equation

$\varphi(\omega) = \varepsilon\omega + c_1,\ \varepsilon = \pm 1, \quad \varphi = 0.$

Solutions of the eikonal equation

$(x_3^2 + u^2)^{1/2} = \varepsilon x_0 + c,\ \varepsilon = \pm 1, \quad x_3^2 + u^2 = 0.$

4. $\left\langle -X_1,\ X_2,\ -L_3 - \frac{\lambda}{2}(P_3 + C_3), 0 < \lambda < 1 \right\rangle :$

Ansatz

$(x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = x_0.$

Reduced equation

$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0.$

Solutions of the reduced equation

$\varphi(\omega) = \varepsilon\omega + c_1,\ \varepsilon = \pm 1, \quad \varphi = 0.$

Solutions of the eikonal equation

$(x_3^2 + u^2)^{1/2} = \varepsilon x_0 + c_1,\ \varepsilon = \pm 1, \quad x_3^2 + u^2 = 0.$

5. $\langle -X_1,\ X_2,\ -(L_3 + \lambda G), \lambda > 0 \rangle :$

Ansatz

$(x_0^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3.$

Reduced equation

$(\varphi' - 1)(\varphi' + 1) = 0.$

Solutions of the reduced equation

$\varphi(\omega) = \varepsilon\omega + c_1,\ \varepsilon = \pm 1.$

Solutions of the eikonal equation

$(x_0^2 - u^2)^{1/2} = \varepsilon x_3 + c_1,\ \varepsilon = \pm 1.$

6. $\langle X_1,\ -X_2,\ -(L_3 + \alpha X_3), \alpha > 0\rangle :$

Ansatz

$u = \varphi(\omega), \quad \omega = x_0.$

Reduced equation

$(\varphi' - 1)(\varphi' + 1) = 0.$

Solutions of the reduced equation

$\varphi(\omega) = \varepsilon\omega + c_1,\ \varepsilon = \pm 1.$

Solutions of the eikonal equation

$u = \varepsilon x_0 + c_1,\ \varepsilon = \pm 1.$

7. $\langle X_1,\ -X_2,\ P_3 - L_3 - 2\alpha X_0, \alpha > 0\rangle :$

Ansatz

$(x_0 + u)^3 + 6\alpha x_3(x_0 + u) + 6\alpha^2(x_0 - u) = \varphi(\omega),$

$\omega = (x_0 + u)^2 + 4x_3\alpha.$

Reduced equation

$4(\varphi')^2 - 9\omega = 0.$

Solutions of the reduced equation

$\varphi(\omega) = \varepsilon\omega^{3/2} + c,\ \varepsilon = \pm 1.$

Solutions of the eikonal equation

$$(x_0+u)^3+6\alpha x_3(x_0+u)+6\alpha^2(x_0-u)=$$
$$=\varepsilon\left((x_0+u)^2+4\alpha x_3\right)^{3/2}+c_1,\ \varepsilon=\pm 1.$$

Reductions to nonlinear ODEs

From the invariants of three nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to nonlinear ODEs.

1. $\left\langle X_1, -X_2, -L_3-\frac{1}{2}\left(P_3+C_3\right)-\alpha(X_0+X_4), \alpha>0\right\rangle:$

Ansatz

$$\alpha\arctan\frac{x_3}{u}-x_0=\varphi(\omega),\quad \omega=(x_3^2+u^2)^{1/2}.$$

Reduced equation

$$\omega^4\left(\omega^2(\varphi')^2-\omega^2+\alpha^2\right)=0.$$

Solutions of the reduced equation

$$\omega=0,\quad \varphi(\omega)=\varepsilon\sqrt{\omega^2-\alpha^2}-i\varepsilon\alpha\mathrm{arctanh}\frac{i\alpha}{\sqrt{\omega^2-\alpha^2}}+$$
$$+c,\ \varepsilon=\pm 1.$$

Solutions of the eikonal equation

$$x_3^2+u^2=0,\quad \alpha\arctan\frac{x_3}{u}-x_0=\varepsilon\sqrt{x_3^2+u^2-\alpha^2}-$$
$$-i\varepsilon\alpha\mathrm{arctanh}\frac{i\alpha}{\sqrt{x_3^2+u^2-\alpha^2}}+c,\ \varepsilon=\pm 1.$$

2. $\Big\langle X_1,\ X_2,\ L_3 + \dfrac{\lambda}{2}(P_3 + C_3) + \alpha(X_0 + X_4),\ \alpha > 0,$
$0 < \lambda < 1\rangle:$

Ansatz
$$\alpha \arctan \frac{x_3}{u} - \lambda x_0 = \varphi(\omega), \quad \omega = (x_3^2 + u^2)^{1/2}.$$

Reduced equation

$$\omega^4 \left(\omega^2(\varphi')^2 - \lambda^2\omega^2 + \alpha^2\right) = 0.$$

Solutions of the reduced equation

$$\omega = 0, \quad \varphi(\omega) = \varepsilon\sqrt{\lambda^2\omega^2 - \alpha^2} -$$
$$-i\varepsilon\alpha\text{arctanh}\frac{i\alpha}{\sqrt{\lambda^2\omega^2 - \alpha^2}} + c,\ \varepsilon = \pm 1.$$

Solutions of the eikonal equation

$$\alpha \arctan \frac{x_3}{u} - \lambda x_0 = \varepsilon\sqrt{\lambda^2(x_3^2 + u^2) - \alpha^2} -$$
$$-i\varepsilon\alpha\text{arctanh}\frac{i\alpha}{\sqrt{\lambda^2(x_3^2 + u^2) - \alpha^2}} + c,\ \varepsilon = \pm 1,$$
$$x_3^2 + u^2 = 0.$$

3. $\langle X_1,\ X_2,\ L_3 + \lambda G + \alpha X_3, \alpha > 0, \lambda > 0\rangle:$

Ansatz
$$\lambda x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = (x_0^2 - u^2)^{1/2}.$$

Reduced equation

$$\omega(\omega(\varphi')^2 + 2\alpha\varphi' - \lambda^2\omega) = 0.$$

Solutions of the reduced equation

$$\omega = 0,\ \varphi(\omega) = \varepsilon\sqrt{\lambda^2\omega^2 + \alpha^2} - \varepsilon\alpha \mathrm{arctanh}\frac{\alpha}{\sqrt{\lambda^2\omega^2 + \alpha^2}} - $$
$$-\alpha \ln(\omega) + c_1,\ \varepsilon = \pm 1.$$

Solutions of the eikonal equation

$$x_0^2 - u^2 = 0,\ \lambda x_3 - \alpha \ln(x_0 + u) = \varepsilon\sqrt{\lambda^2(x_0^2 - u^2) + \alpha^2} -$$
$$-\varepsilon\alpha \mathrm{arctanh}\frac{\alpha}{\sqrt{\lambda^2(x_0^2 - u^2) + \alpha^2}} - \frac{\alpha}{2}\ln(x_0^2 - u^2) + c_1,$$
$$\varepsilon = \pm 1.$$

It should be noted that subalgebras (1)–(3) do not belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

Reductions to PDEs

From the invariants of two nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to PDEs.

1. $\langle X_1,\ X_2,\ L_3\rangle :$

   Ansatz

   $u = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0,\ \omega_2 = x_3.$

   Reduced equation

   $\varphi_1^2 - \varphi_2^2 - 1 = 0.$

   Solutions of the reduced equation

   $\varphi(\omega_1, \omega_2) = \varepsilon\sqrt{c_2^2 + 1}\,\omega_1 + c_2\omega_2 + c_1 + c_2, \varepsilon = \pm 1.$

Solutions of the eikonal equation

$u = \varepsilon\sqrt{c_2^2+1}\,x_0 + c_2 x_3 + c_1 + c_2, \varepsilon = \pm 1.$

2. $\langle P_1,\ P_2,\ L_3\rangle$:

Ansatz

$x_3 = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 + u,\ \omega_2 = x_0^2 - x_1^2 - x_2^2 - u^2.$

The reduced equation

$4\omega_2\varphi_2^2 + 4\omega_1\varphi_1\varphi_2 - 1 = 0.$

Solutions of the reduced equation

$\varphi(\omega_1, \omega_2) = c_1 \ln \omega_1 - \varepsilon(\omega_2 + c_1^2)^{1/2} +$

$+ i\varepsilon c_1 \arctan \dfrac{\sqrt{\omega_2 + c_1^2}}{ic_1} - \dfrac{c_1}{2} \ln \omega_2 + c_2,\ \varepsilon = \pm 1.$

Solutions of the eikonal equation

$x_3 = c_1 \ln(x_0 + u) - \varepsilon(x_0^2 - x_1^2 - x_2^2 - u^2 + c_1^2)^{1/2} +$

$+ i\varepsilon c_1 \arctan \dfrac{\sqrt{x_0^2 - x_1^2 - x_2^2 - u^2 + c_1^2}}{ic_1} -$

$-\dfrac{c_1}{2} \ln(x_0^2 - x_1^2 - x_2^2 - u^2) + c_2, \varepsilon = \pm 1.$

As we see, in the above two cases, the reduced equations are PDEs. The reason is that the subalgebras corresponding to them have rank 2. Therefore, they have three invariants. As a rule, the ansatzes, which can be constructed with the help of those invariants, reduce the eikonal equation to PDEs.

It should be noted that subalgebras (1) and (2) belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

There are no reductions

From the invariants of the remaining two nonconjugate subalgebras it is impossible to construct the ansatzes, which reduce the eikonal equation. The details on this theme can be found in Chapter 2.

3.3.9 Lie Algebras of the Type $A_{3,7}^a$

In Chapter 2, we have presented two ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,7}^a$.

By now, we performed the symmetry reduction of the eikonal equation to differential equations with a fewer number of independent variables using those ansatzes. Some classes of invariant solutions are constructed.

Below, we present the results obtained.

Reductions to equations, which can be splitted on two linear ODEs

Taking into account the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to that, which could be splitted on two linear ODEs.

$\langle P_1,\ P_2,\ L_3 + \lambda G, \lambda > 0\rangle :$

Ansatz

$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3.$

Reduced equation

$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0.$

Solutions of the reduced equation

$\varphi(\omega) = \varepsilon\omega + c_1,\ \varepsilon = \pm 1, \quad \varphi = 0.$

Solutions of the eikonal equation

$(x_0^2-x_1^2-x_2^2-u^2)^{1/2} = \varepsilon x_3+c_1,\ \varepsilon = \pm 1, \quad x_0^2-x_1^2-x_2^2-u^2 = 0.$

It should be noted that the subalgebra do not belongs to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

Reductions to nonlinear ODEs

From the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to nonlinear ODE.

$\langle P_1,\ P_2,\ L_3 + \lambda G + \alpha X_3, \alpha > 0,\ \lambda > 0\rangle :$

Ansatz

$$\lambda x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_0^2 - x_1^2 - x_2^2 - u^2.$$

The reduced equation

$$4\omega\,(\varphi')^2 + 4\alpha\varphi' - \lambda^2 = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = \varepsilon(\lambda^2\omega + \alpha^2)^{1/2} - i\varepsilon\alpha \arctan\frac{\sqrt{\lambda^2\omega + \alpha^2}}{i\alpha} - \frac{\alpha}{2}\ln\omega + $$
$$+c, \quad \varepsilon = \pm 1.$$

Solutions of the eikonal equation

$$\lambda x_3 - \alpha \ln(x_0 + u) = \varepsilon(\lambda^2(x_0^2 - x_1^2 - x_2^2 - u^2) + \alpha^2)^{1/2} -$$
$$-i\varepsilon\alpha \arctan\frac{\sqrt{\lambda^2(x_0^2 - x_1^2 - x_2^2 - u^2) + \alpha^2}}{i\alpha} -$$
$$-\frac{\alpha}{2}\ln(x_0^2 - x_1^2 - x_2^2 - u^2) + c,\ \varepsilon = \pm 1.$$

It should be noted that the subalgebra do not belongs to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

3.3.10 Lie Algebras of the Type $A_{3,8}$

In Chapter 2, we have presented only one ansatz, which is invariant with respect to three-dimensional nonconjugate subalgebra of the type $A_{3,8}$.

By now, we performed the symmetry reduction of the eikonal equation to differential equation with a fewer number of independent variables using this ansatz. Some invariant solutions are constructed.

Below, we present the results obtained.

Reductions to PDEs

From the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to PDE.

$\langle P_3,\ G,\ -C_3\rangle :$

Ansatz

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_1,\ \omega_2 = x_2.$$

Reduced equation

$$\varphi^2(\varphi_1^2 + \varphi_2^2 - 1) = 0.$$

Solutions of the reduced equation

$$\varphi(\omega_1, \omega_2) = 0, \quad \varphi(\omega_1, \omega_2) = \varepsilon\sqrt{1 - c_2^2}\,\omega_1 + c_2\omega_2 + c_1,$$

$\varepsilon = \pm 1.$

Solutions of the eikonal equation

$x_0^2 - x_3^2 - u^2 = 0,\ (x_0^2 - x_3^2 - u^2)^{1/2} = \varepsilon(1 - c_2^2)^{1/2} x_1 + c_2 x_2 +$
$+ c_1,\ \varepsilon = \pm 1.$

As we see, the reduced equation is PDEs. As above, the reason is that the corresponding subalgebra has rank 2.

It should be noted that the subalgebra do not belongs to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

3.3.11 Lie Algebras of the Type $A_{3,9}$

In Chapter 2, we have presented two ansatzes, which are invariant with respect to three-dimensional nonconjugate subalgebras of the type $A_{3,9}$.

By now, we performed the symmetry reduction of the eikonal equation to differential equations with a fewer number of independent variables using those ansatzes. Some classes of invariant solutions are constructed.

Below, we present the results obtained.

Reductions to equations, which can be splitted on two linear ODEs

Taking into account the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to that, which could be splitted on two linear ODEs.

$$\left\langle -\frac{1}{2}\left(L_3 + \frac{1}{2}\left(P_3 + C_3\right)\right),\ \frac{1}{2}\left(L_2 + \frac{1}{2}\left(P_2 + C_2\right)\right),\right.$$
$$\left.\frac{1}{2}\left(L_1 + \frac{1}{2}\left(P_1 + C_1\right)\right)\right\rangle :$$

Ansatz

$$(x_1^2 + x_2^2 + x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = x_0.$$

Reduced equation

$$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = \varepsilon\omega + c_1, \ \varepsilon = \pm 1, \quad \varphi = 0.$$

Solutions of the eikonal equation

$$(x_1^2+x_2^2+x_3^2+u^2)^{1/2} = \varepsilon x_0 + c_1, \ \varepsilon = \pm 1, \quad x_1^2+x_2^2+x_3^2+u^2 = 0.$$

It should be noted that the subalgebra do not belongs to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

Reductions to PDEs

From the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to PDE.

$\langle -L_3, \ -L_2, \ -L_1 \rangle$:

Ansatz

$$u = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0, \ \omega_2 = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

Reduced equation

$$\varphi_1^2 - \varphi_2^2 - 1 = 0.$$

Solutions of the reduced equation

$$\varphi(\omega_1, \omega_2) = \varepsilon\sqrt{c_2^2 + 1}\,\omega_1 + c_2\omega_2 + c_1, \ \varepsilon = \pm 1.$$

Solutions of the eikonal equation

$$u = \varepsilon(c_2^2 + 1)^{1/2} x_0 + c_2(x_1^2 + x_2^2 + x_3^2)^{1/2} + c_1, \ \varepsilon = \pm 1.$$

As we see, the reduced equation is PDEs. As above, the reason is that the corresponding subalgebra has rank 2.

It should be noted that the subalgebra belongs to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

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